Chapter 6
The Chern-Simons electro magnetic spin density
–
from my book:
Understanding Relativistic Quantum Field Theory

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Contents

Chapter 6

The Chern-Simons electro magnetic spin

6.1 Electromagnetic Spin and the axial anomaly

The axial anomaly of the electron was discovered around 1969 by S. L. Adler, John S. Bell and R. Yackiw [1, 2, 3, 4, 5]. It was found that the axial current J_A^μ of the electron (its spin) is not conserved independently. In order to conserve the spin and to keep electromagnetism as a local gauge theory, it is required by quantum perturbation theory that:

$$\partial_\mu j_A^\mu \quad = \quad -\frac{\alpha}{2\pi}\,\partial_\mu C^\mu \tag{6.1}$$

See the chapters on the Dirac equation. The rightmost term of the equation, the Chern-Pontryagin density $\mathcal{A} = \partial_\mu C^\mu$, is non-zero outside the electron's wave function where the charge/current density is zero.

$$\partial_\mu C^\mu \quad = \quad \mathcal{A} \quad = \quad \epsilon_o c\,\tfrac{1}{4}\varepsilon^{\,\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad = \quad -\,2\epsilon_o\,\mathsf{E}\cdot\mathsf{B} \tag{6.2}$$

rom the last term we can derive what can be interpreted as the electromagnetic spin density, [2, 6], the Chern Simons current $\mathcal{C}^\mu$.

$$\mathcal{C}^\mu \quad = \quad \epsilon_o\,\tfrac{1}{2}\varepsilon^{\,\mu\nu\alpha\beta}F_{\alpha\beta}A_\nu \quad = \quad \epsilon_o\,\varepsilon^{\,\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma \tag{6.3}$$

$$\mathcal{C}^\mu \quad = \quad \begin{pmatrix} 0 & -\frac{1}{c}\,\mathsf{H}_x & -\frac{1}{c}\,\mathsf{H}_y & -\frac{1}{c}\,\mathsf{H}_z \\ \frac{1}{c}\,\mathsf{H}_x & 0 & \mathsf{D}_z & -\,\mathsf{D}_y \\ \frac{1}{c}\,\mathsf{H}_y & -\,\mathsf{D}_z & 0 & \mathsf{D}_x \\ \frac{1}{c}\,\mathsf{H}_z & \mathsf{D}_y & -\,\mathsf{D}_x & 0 \end{pmatrix} \begin{pmatrix} A_0 \\ -A_x \\ -A_y \\ -A_z \end{pmatrix} \tag{6.4}$$

This is a four-vector field which we can write down explicitly as a 3d vector and a time-component. (with $\Phi = cA^0$)

$$\text{EM Spin Density:} \qquad \vec{\mathcal{C}} \;=\; \mathsf{D}\times\vec{A} \;+\; \mathsf{H}\,\Phi\;, \qquad \mathcal{C}^o \;=\; \tfrac{1}{c}\,\mathsf{H}\cdot\vec{A} \tag{6.5}$$

The term $\epsilon_o\mathsf{E}\times\vec{A}$ is for instance discussed in Mandel and Wolf [7] as the *electromagnetic intrinsic or spin density.* This term by itself is however not Lorentz invariant, a requirement which leads to the full expression given above. The spin density $\mathcal{C}^\mu$ transforms relativistically correct like a contravariant vector, as a spin vector should. We will, throughout this document, use the Liènard Wiechert potentials which are arguably the appropriate choice for "physical" potential fields.

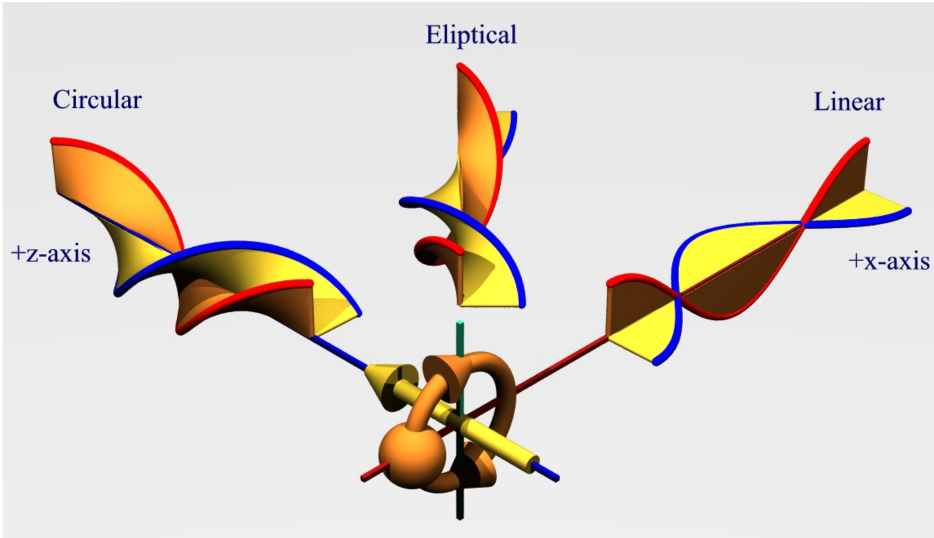

Figure 6.1: Polarization depending on the direction in the xz-plane

Figure 6.1 recalls the dependency of the polarization type on the angle between the source spin (the spin-1 transition of an atomic orbital), and the direction of the photon propagation. The quantum mechanical transition current causing the electromagnetic radiation is the interference between the initial and final state of the emitting atom.

We will show that for *any* type of polarization the following relations hold locally at *each* point of the field:

$$\frac{|\ \vec{\mathcal{C}}\ |}{\mathcal{E}_{max}} = \frac{|\ \text{spin density}\ |}{|\ \text{energy density}\ |} = \frac{\hbar}{E_{max}}\ \cos\phi \tag{6.6}$$

$$\frac{|\ \vec{\mathcal{C}}\ |}{|\ \vec{\mathcal{P}}_{max}\ |} = \frac{|\ \text{spin density}\ |}{|\ \text{momentum density}\ |} = \frac{\hbar}{p_{max}}\ \cos\phi \tag{6.7}$$

We see that the spin density has the required value $\hbar$ (spin=1). The spin direction is *always* along the line of propagation as required by a massless spin 1 field. The factor $\cos\theta$ is the projection of the spin on the line of propagation. The spin density propagation is *time independent*, unlike the energy density and momentum density $\vec{\mathcal{P}}$ which vary in time between a minimum and a maximum for any polarization other than circular polarization. We can set up a table. (with $\mathsf{D} = \epsilon_o\mathsf{E}$ and $\mu_o\mathsf{H} = \mathsf{B}$)

Electromagnetic Spin Density components

polarization	orbit spin	EM spin $\vec{C}$	$\mathsf{D} \times \vec{A}$	$\mathsf{H}\ \Phi$
Linear	$+\hbar$	0	$+\hbar$	$-\hbar$
Linear	$-\hbar$	0	$-\hbar$	$+\hbar$
Circular	$+\hbar$	$+\hbar$	$+\hbar$	0
Circular	$-\hbar$	$-\hbar$	$-\hbar$	0

(6.8)

The table shows the decomposition of $\vec{C}$. The two subcomponents $\mathsf{D} \times \vec{A}$ and $\mathsf{H}\Phi$ are non-zero for linear polarized photons even though the total spin is zero due to the $\cos\theta$. We can write for the absolute values of the subcomponents in case of linear polarized radiation:

Linear polarized radiation

$$\frac{|\ \mathsf{D} \times \vec{A}\ |}{|\quad \mathcal{E} \quad|} = \frac{|\ \mathsf{H}\Phi\ |}{|\ \mathcal{E}\ |} = \frac{|\quad \text{spin density} \quad|}{|\quad \text{energy density} \quad|} = \frac{\hbar}{E} \tag{6.9}$$

$$\frac{|\ \mathsf{D} \times \vec{A}\ |}{|\quad \vec{\mathcal{P}} \quad|} = \frac{|\ \mathsf{H}\Phi\ |}{|\ \vec{\mathcal{P}}\ |} = \frac{|\quad \text{spin density} \quad|}{|\ \text{momentum density}\ |} = \frac{\hbar}{p} \tag{6.10}$$

The subcomponents *do* vary in time, and for linear polarization they vary in the same way as the energy and momentum densities. Looking for further confirmation of $\mathcal{C}^\mu$ as the spin density will study it for the fields of a charged electron at rest. We will find that, further away from the electron's wave function, the following relation holds, with g_e as the gyromagnetic ratio (2.00223..)

The static electron fields

$$\frac{|\ \vec{\mathcal{C}}\ |}{\mathcal{E}} = \frac{|\quad \text{spin density} \quad|}{|\ \text{energy density}\ \ |} = g_e \frac{\hbar}{E}\ \cos\theta \tag{6.11}$$

The electromagnetic spin density points always in the direction of propagation as required and the factor $\cos\theta$ is again the result of the projection of the electron's spin on the line of propagation. Remarkably, this equation fixes the relation between the electric charge and magnetic moment of the electron and other charged leptons.

6.2 The subcomponents of the EM spin density

Figure 6.2 shows the two subcomponents for the different types of polarization. The upper image shows the $\mathsf{E} \times \vec{A}$ component. For circular polarization this component is always $\hbar$. (The vectors of equal length overlap each other in the image)

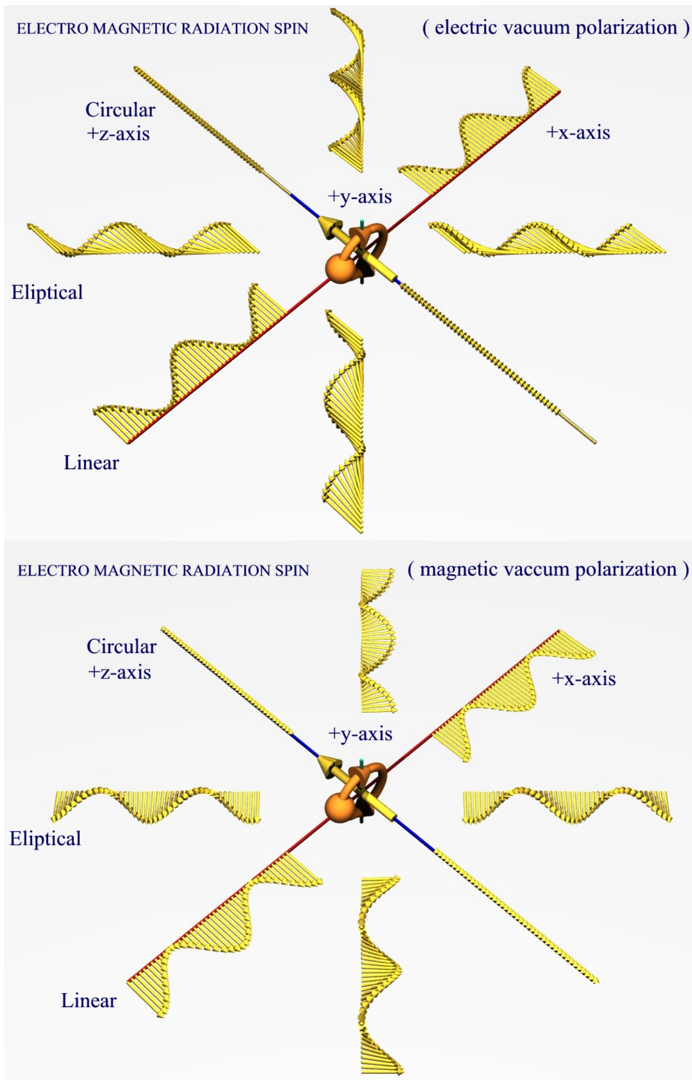

Figure 6.2: Radiation spin density from elec/magnetic polarization

In the linear polarization case the vectors point all in the same direction but the *length* varies in time exactly like the the energy and momentum densities. The lower image shows the $\mathsf{B}\Phi$ term. It is zero for circular polarization but it is the the opposite of $\mathsf{E}\times\vec{A}$ in case of linear polarization. In this paper we will present some arguments that $\mathsf{E}\times\vec{A}$ is the spin density from *electric* vacuum polarization while $\mathsf{B}\Phi$ can be regarded as the spin density associated with *magnetic* vacuum polarization.

The 0-component of a spin vector in general should be zero in the spin's rest frame. Indeed, the 0-component of $\mathcal{C}^\mu$ turns out to be zero in both cases if the source is at rest. That is, in case of the static fields of the electron as well as in the case of a spin-1 transition current of an atomic orbital. In the latter case there is an effective orbiting charge density, however, the ratio of velocity and acceleration of this charge density is so that $\mathcal{C}^o = 0$ in the center's rest frame.

6.3 Total angular momentum from spin

A gradient in the spin density also contributes (independently) to the total angular momentum. This is a result of Stokes law which is illustrated in figure **??**. It shows a varying spin density in the form of a plane wave pattern.

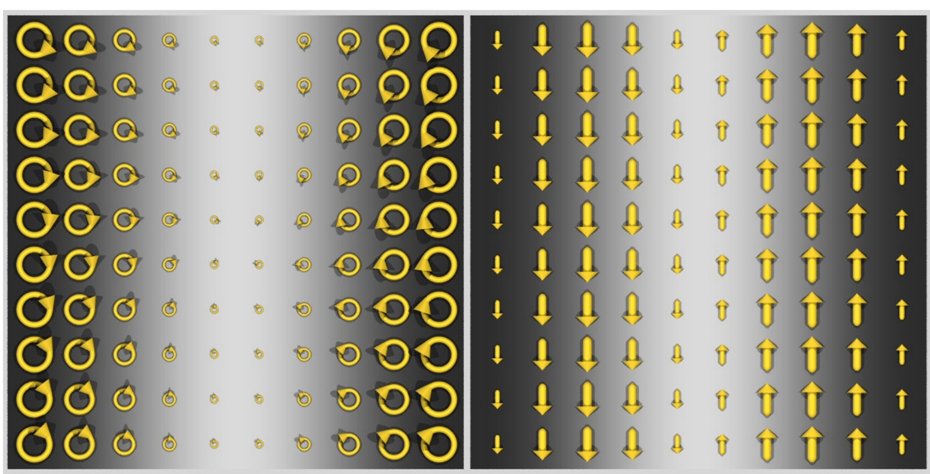

Figure 6.3: Spin density and effective current density

The circular currents cancel each other if the spin density is constant. A gradient however gives rise to an effective current according to Stokes law as illustrated in the right image. In the rest frame ($\mathcal{C}^o = 0$) we can express this effective current as the curl of the spin density.

$$\vec{\mathcal{P}}_s \ = \nabla \times \vec{\mathcal{C}} \tag{6.12}$$

This translates into an angular momentum density $\vec{\mathcal{L}}_s$

$$\vec{\mathcal{L}}_s \ = \vec{R} \times (\nabla \times \vec{\mathcal{C}}) \tag{6.13}$$

Which has to be added to the total angular momentum density $\vec{\mathcal{J}}_s$

$$\vec{\mathcal{J}}_s \ = \vec{\mathcal{L}}_s \ + \ \vec{\mathcal{C}} \tag{6.14}$$

We will derive these fields for both the spin-1 transition radiation and the static electron fields. It turns out that in case of the radiation the following holds for each point of the radiation field.

Arbitrary polarized radiation from spin-1 transition:

$$\frac{|\ \vec{\mathcal{J}}\ |}{\mathcal{E}_{max}} = \frac{|\ \text{spin + angular mom. density}\ |}{|\ \ \text{energy density}\ \ |} = \frac{\hbar}{E} \tag{6.15}$$

$$\frac{|\ \vec{\mathcal{J}}\ |}{|\ \vec{\mathcal{P}}_{max}\ |} = \frac{|\ \text{spin + angular mom. density}\ |}{|\ \text{momentum density}\ |} = \frac{\hbar}{p} \tag{6.16}$$

The direction of $\vec{\mathcal{J}}$ is always in the *same* direction as the spin of the source. Similarly, For each point further away from the electron's wave function the ratio between the total angular momentum density and the energy density is:

The static electron fields

$$\frac{|\ \vec{\mathcal{J}}\ |}{\mathcal{E}} = \frac{|\ \text{spin + angular mom. density}\ |}{|\ \ \text{energy density}\ \ |} = g_e\frac{\hbar}{E} \tag{6.17}$$

The direction of $\vec{\mathcal{J}}$ is again that of the source spin. The fields for the total angular momenta are *independent* of the angle θ between the spin of the source and the direction of propagation. The angle dependency arises when we split the total angular momentum into its components.

$$\vec{\mathcal{J}}_s \;=\; \vec{\mathcal{L}}_s + \vec{\mathcal{C}} \;=\; \hbar \sin\theta\ \hat{\mathbf{r}}_\perp +\ \hbar\cos\theta\ \hat{\mathbf{r}}_\parallel \tag{6.18}$$

The spin density $\vec{\mathcal{C}}$ must per definition be in the direction of the propagation since the electromagnetic field is a massless field. The occurrence of the gyromagnetic ratio g_e in (6.17) means that the equation is valid for any charged particle with an arbitrary magnetic moment

6.4 Spin density from electric vacuum polarization

Before we start the actual derivations of the various fields we want to briefly discuss the arguments why $\mathsf{E}\times A$ can be associated with the electric vacuum polarization and $\mathsf{B}\Phi$ with the magnetic vacuum polarization. The expressions for the spin and spin density of the electromagnetic field to be associated with electric vacuum polarization are respectively:

$$\vec{s}_{elec} \;=\; \epsilon_o \int dx^3\ (\ \mathsf{E}\times A\ ), \qquad \vec{\mathcal{C}}_{elec} \;=\; \epsilon_o(\ \mathsf{E}\times A\ ) \tag{6.19}$$

These expressions are most easily understood in the *"vacuum polarization"* or *"displacement current"* picture of the electric field E. The electric field is proportional to the polarization of the vacuum due to a displacement of opposite virtual charges in the direction of the field. The component of the vector potential A transversal to the field E now modifies the momenta of the opposite charges in opposite directions, thereby creating an intrinsic spin of the electromagnetic field.

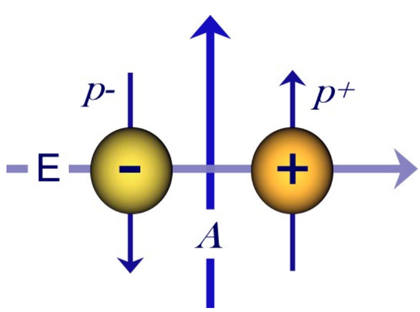

Figure 6.4: Spin from electric vacuum polarization: $\mathsf{D}\times A$

Recall that the actual speed of the charges is determined by the subtraction of the total (canonical) momentum, with the momentum due to the vector potential.

6.5 Spin density from magnetic vacuum polarization

The expressions for the spin and spin density which we would like to associate with the magnetic vacuum polarization are respectively:

$$\vec{s}_{magn} \;=\; \frac{1}{\mu_o}\int dx^3\ (\ \mathsf{B}\ \Phi\ ), \qquad \vec{\mathcal{C}}_{magn} \;=\; \frac{1}{\mu_o}(\ \mathsf{B}\ \Phi\ ) \tag{6.20}$$

We assume that the neutral vacuum can be polarized by vacuum fluctuations involving pairs of opposite virtual charges with anti-parallel spin and parallel magnetic moment. Figure 6.5 shows the inertial spins at the left and the magnetic moments at the right in case of a *theoretically maximally* polarized vacuum. In reality one expects only very small systematic deviations from random directions.

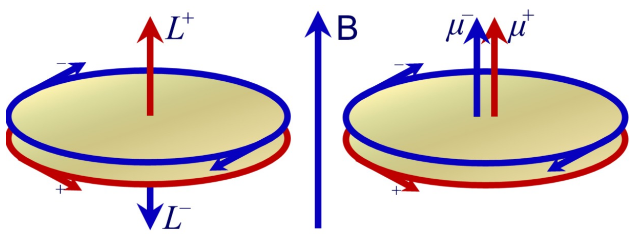

Figure 6.5: Spin from magnetic vacuum polarization: $\mathsf{H}\Phi$

The magnetization is proportional to B. The net inertial spin is zero as long as Φ is zero. A non-zero potential field Φ will shift the canonical energies of the particle and anti-particles in opposite ways and the two inertial spins do not cancel each other anymore. The result is a non-zero net spin density.

6.6 The static electron's spin density fields

We assume the Pauli-Weisskopf interpretation of the wave function as a distributed charge and spin density. The electromagnetic potentials and fields are the convolutions of the probability density with the potentials and fields of the point charge and spin. Assuming that the spin is up in the z-direction we have.

$$\Phi = \frac{q}{4\pi\epsilon_0 r} \tag{6.21}$$

$$\vec{A} = \frac{\mu_o\mu_e}{4\pi r^2}\left(-\frac{y}{r}, \quad \frac{x}{r}, \quad 0 \right) \tag{6.22}$$

$$\mathsf{E} = \frac{q}{4\pi\epsilon_o r^2}\left(\frac{x}{r}, \quad \frac{y}{r}, \quad \frac{z}{r} \right) \tag{6.23}$$

$$\mathsf{B} = \frac{\mu_o\mu_e}{4\pi r^3}\left(3\,\frac{xz}{r^2}, \quad 3\,\frac{yz}{r^2}, \quad 3\,\frac{zz}{r^2}-1 \right) \tag{6.24}$$

Where the dimensionless quantities between brackets determine the x, y and z-components. We will use these fields to determine the Chern Simons point spin as well as the derived angular momentum densities.

$$\vec{\mathcal{C}} = \mathsf{D}\times\vec{A} + \mathsf{H}\,\Phi\,, \qquad \mathcal{C}^o = \frac{1}{c}\mathsf{H}\cdot\vec{A} \tag{6.25}$$

The time component of the spin $\mathcal{C}^o$ is 0 in the rest frame as it should be. For the spin vector $\vec{\mathcal{C}}$ and its components we find.

$$\mathsf{D}\times\vec{A} = \frac{q\mu_o\mu_e}{16\pi^2 r^4}\left(-\frac{xz}{r^2}, \quad -\frac{yz}{r^2}, \quad \frac{x^2+y^2}{r^2} \right) \tag{6.26}$$

$$\mathsf{H}\Phi = \frac{q\mu_o\mu_e}{16\pi^2 r^4}\left(3\,\frac{xz}{r^2}, \quad 3\,\frac{yz}{r^2}, \quad 3\,\frac{zz}{r^2}-1 \right) \tag{6.27}$$

$$\vec{\mathcal{C}} = \frac{q\mu_0\mu_e}{8\pi^2 r^4}\left(\frac{xz}{r^2}, \quad \frac{yz}{r^2}, \quad \frac{zz}{r^2} \right) \tag{6.28}$$

For the total angular momentum we derive the effective current $\vec{P}_s$ from the curl of the spin-density $\vec{C}$. So that we can calculate the effective angular momentum $\vec{L}_s$. The latter added to the spin density gives the total angular momentum due to spin.

$$\vec{\mathcal{P}}_s = \nabla\times\vec{C}, \qquad \vec{L}_s = \vec{R}\times(\nabla\times\vec{C}), \qquad \vec{J}_s = \vec{L}_s + \vec{C} \tag{6.29}$$

$$\vec{\mathcal{P}}_s = \frac{q\mu_0\mu_e}{8\pi^2 r^5}\left(-\frac{y}{r}, \quad \frac{x}{r}, \quad 0 \right) \tag{6.30}$$

$$\vec{\mathcal{L}}_s = \frac{q\mu_0\mu_e}{8\pi^2 r^4}\left(-\frac{xz}{r^2}, \quad -\frac{yz}{r^2}, \quad \frac{x^2+y^2}{r^2} \right) \tag{6.31}$$

$$\vec{\mathcal{J}}_s = \frac{q\mu_0\mu_e}{8\pi^2 r^4}\left(0, \quad 0, \quad 1 \right) \tag{6.32}$$

Figure 6.6 shows the two components of the spin-density in the y-z plane. The left image shows the electric vacuum polarization component $\epsilon_o \mathsf{E} \times \vec{A}$ while the right image shows the component $\frac{1}{\mu_o}\mathsf{B}\Phi$ due to magnetic vacuum polarization.

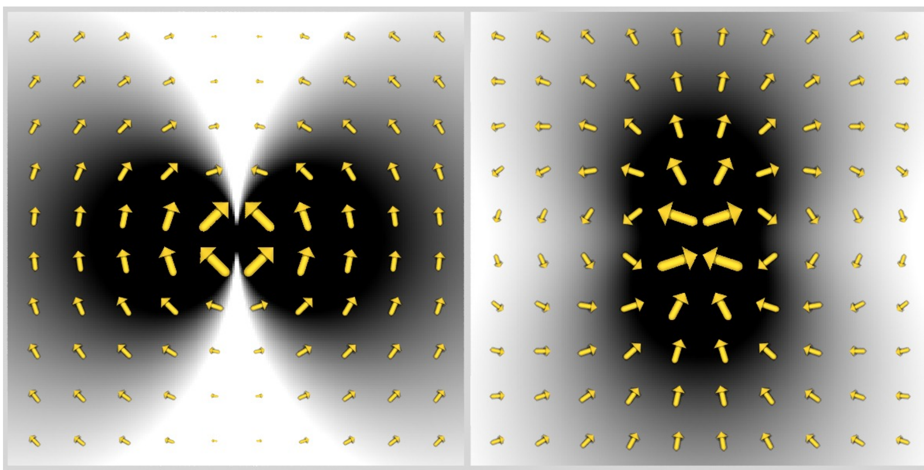

Figure 6.6: Spin density from electric (l) and magnetic (r) vacuum pol.

Figure 6.7 shows the spin density $\vec{\mathcal{C}}$ in the left image. The spin density is always pointed away from the source. The magnitude of the spin is maximal on the vertical z-axis.

The right image of figure 6.7 shows the total spin plus angular momentum density. It always points in the same direction as the spin of the source. The spin density on the left can be viewed as the projection of the total angular momentum density on the direction of the propagation ($\vec{r}$) away from the source. The spin density of massless fields has to point along the line of propagation.

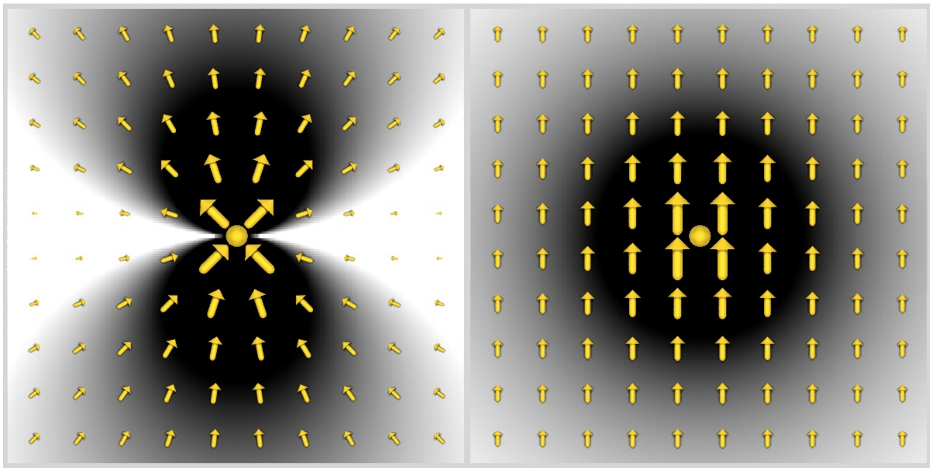

Figure 6.7: Spin density (l) and Total spin plus angular momentum (r)

6.7 Spin density versus energy/momentum density

For the energy of the electro static and magneto static fields of the static electron we obtain respectively.

$$\text{Energy density in } J/m^3: \qquad \mathcal{E} \;=\; \frac{1}{2}\left(\epsilon_o \mathsf{E}^2 + \frac{1}{\mu_o}\mathsf{B}^2\right) \tag{6.33}$$

$$\mathcal{E}_{elec} \;=\; \frac{q^2}{32\pi^2\epsilon_o r^4}, \qquad \mathcal{E}_{magn} \;=\; \frac{\mu_o\mu_e^2}{32\pi^2 r^6}\left(\frac{3z^2}{r^2}+1\right) \tag{6.34}$$

We see that the magnetic energy decrease much faster when r increases. In order to get a qualitative feeling for these energy densities we integrate them to total energies using a cut-off radius of r_o.

$$\text{Electric energy:} \quad \frac{q^2}{8\pi\epsilon_o r_o}, \qquad \text{Magnetic energy:} \quad \frac{\mu_o\mu_e^2}{12\pi r_o^3} \tag{6.35}$$

If the cut-off radius r_o is half the classical electron radius then the energy of the field becomes equal to the rest mass of the electron: $r_o = r_e/2 = 1.4089701625\ 10^{-15}\ m$. The rest mass energy is reached sooner if we consider the magneto static energy associated with the magnetic moment, (at $r_o = 3.27413591\ 10^{-14}\ m$)

Total field energy relative to the rest mass

	Bohr radius (r_e/α^2)	Compton radius (r_e/α)	Electron radius (r_e)
E_{elec}	2.662567 e-5	3.648676 e-3	0.5000000000
E_{magn}	2.368572 e-10	6.095238 e-4	1.568536 e+3

However further away, outside the electron's wave function, we can ignore the magneto static contribution in single particle systems because of the $1/r^3$ decay. In this case the ratio between the total spin and angular momentum density and the energy density becomes:

$$\left| \frac{q\mu_0\mu_e}{8\pi^2 r^4} \right| \div \left| \frac{q^2}{32\pi^2\epsilon_o r^4} \right| = \frac{4}{c^2}\,\frac{\mu_e}{q} = g_e\frac{\hbar}{mc^2} = g_e\frac{\hbar}{E} \tag{6.36}$$

The expression for the spin density (6.37) can be written as.

$$\vec{\mathcal{C}} = \frac{q\mu_0\mu_e}{8\pi^2 r^4}\;\hat{r}\cos\theta, \qquad \left(\cos\theta = \frac{z}{r}\right) \tag{6.37}$$

Which explicitly shows that the spin is transversely polarized along $\vec{r}$ static field energy. While the $\cos\theta$ represents the projection of the source spin on the line of propagation.

We calculated $\vec{\mathcal{P}}_s$, the effective momentum from the curl of the spin density. It is interesting to compare this result with the momentum density represented by the ordinary Poynting vector. The Poynting vector represents the energy-flux in Joule going through one unit of area during one unit of time, to obtain the effective momentum density $\mathcal{P}$ expressed in Js/m^4, that is, momentum Js/m per unit of volume $1/m^3$ one must multiply by c^2, so:

$$\text{Momentum density in } Js/m^4: \qquad \vec{\mathcal{P}} = \epsilon_o\;\mathsf{E}\times\mathsf{B} \tag{6.38}$$

$$\vec{\mathcal{P}} = \frac{q\mu_o\mu_e}{16\pi^2\epsilon_o r^5}\left(-\frac{y}{r},\;\frac{x}{r},0\right) = \frac{1}{2}\,\vec{\mathcal{P}}_s \tag{6.39}$$

This momentum has the same x, y and z-components (up to a factor 2), as the momentum obtained from the spin. We can interpret this also as an angular momentum, which consequently is half of the angular momentum $\vec{\mathcal{L}}_s$ originating from the electromagnetic spin. We obtain as $\vec{\mathcal{L}} = \vec{r} \times \vec{\mathcal{P}}$, expressed in Js/m^3.

$$\text{Ang.mom.density:} \quad \vec{\mathcal{L}} \;=\; \frac{q\mu_e}{16\pi^2\epsilon_o c^2} \left\{ \frac{-xz}{r^6},\ \frac{-yz}{r^6},\ \frac{x^2+y^2}{r^6} \right\} \;=\; \frac{1}{2}\,\vec{\mathcal{L}}_s \tag{6.40}$$

For convenance we list all the fields written as vector expressions. The basic electromagnetic potentials and fields are given by.

$$\Phi \;=\; \frac{q}{4\pi\epsilon_0 r}, \qquad \mathsf{E} \;=\; \frac{q}{4\pi\epsilon_o r^2}\,\hat{r} \tag{6.41}$$

$$\vec{A} \;=\; \frac{\mu_o\mu_e}{4\pi r^2}\,(\,\hat{\mu}_e \times \hat{r}\,), \qquad \mathsf{B} \;=\; \frac{\mu_o\mu_e}{4\pi r^3}\,(\,3\,\hat{r}\,(\,\hat{\mu}_e\cdot\hat{r}\,) - \hat{\mu}_e\,) \tag{6.42}$$

The spin and angular momentum fields as vector expressions:

$$\mathsf{D}\times\vec{A} \;=\; \frac{q\mu_o\mu_e}{16\pi^2 r^4}\,(-\hat{r}(\,\hat{\mu}_e\cdot\hat{r}\,) + \hat{\mu}_e\,) \tag{6.43}$$

$$\mathsf{H}\Phi \;=\; \frac{q\mu_o\mu_e}{16\pi^2 r^4}\,(\,3\,\hat{r}\,(\,\hat{\mu}_e\cdot\hat{r}\,) - \hat{\mu}_e\,) \tag{6.44}$$

$$\vec{\mathcal{C}} \;=\; \frac{q\mu_0\mu_e}{8\pi^2 r^4}\;\hat{r}\,(\,\hat{\mu}_e\cdot\hat{r}\,) \tag{6.45}$$

$$\vec{\mathcal{P}}_s \;=\; \frac{q\mu_0\mu_e}{8\pi^2 r^5}\,(\,\hat{\mu}_e\times\hat{r}\,) \tag{6.46}$$

$$\vec{\mathcal{L}}_s \;=\; \frac{q\mu_0\mu_e}{8\pi^2 r^4}\,(-\hat{r}(\,\hat{\mu}_e\cdot\hat{r}\,) + \hat{\mu}_e\,) \tag{6.47}$$

$$\vec{\mathcal{J}}_s \;=\; \frac{q\mu_0\mu_e}{8\pi^2 r^4}\;\hat{\mu}_e \tag{6.48}$$

6.8 The atomic source of the spin 1 radiation field

We express the initial state ψ^i and final state ψ^f split in spherical components in order to explicitly show the z-component of the orbital angular momentum. The transition current is given by the interference between the two when they are in a superposition $\psi = a\psi^i + b\psi^f$

$$\psi_i = \psi_i^r\ \psi_i^\theta\ \psi_i^\phi = \psi_i^r\ \psi_i^\theta\ e^{m_i\phi}, \qquad \psi_f = \psi_f^r\ \psi_f^\theta\ \psi_f^\phi = \psi_f^r\ \psi_f^\theta\ e^{m_f\phi} \tag{6.49}$$

$$j_{if}^\mu \ = \ ab\left(\bar{\psi}_f\gamma^\mu\psi_i \ + \ \bar{\psi}_i\gamma^\mu\psi_f\right) \tag{6.50}$$

The interference part of the superposition will have terms for the charge- and current density which contain $\cos((m_i - m_f)\phi)$ where $(m_i - m_f) = \pm 1$. The associated current can be represented by an orbiting charge, at a radius which can be neglected at a range far enough away, where we also can approximate the spherical waves with plane waves. We can write for transition current:

$$\vec{v} = \{\ v\cos(\phi),\ v\sin(\phi),\ 0\ \}, \qquad \vec{a} = \{\ -v\omega\sin(\phi),\ v\omega\cos(\phi),\ 0\ \} \tag{6.51}$$

$$\vec{\beta} = \frac{\vec{v}}{c}, \qquad \phi = \omega t - kr, \qquad \omega = ck = \frac{E}{\hbar} \tag{6.52}$$

Where $\vec{v}$ and $\vec{a}$ are the speed and acceleration of the radiating charge density at the time when the emission occurred. The physical important relation is the ratio $a/v = \omega$ and the fact that a and v are always orthogonal to each other.

6.9 Application of the Liénard Wiechert potentials

We recall equations (**??**), (**??**) and (**??**) here for the Liénard Wiechert potentials and fields. We only take the radiating parts of the E and B fields as we may do so at a sufficiently large distance away from the source. The radiation terms decay only with $1/r$ while the non-radiating terms decay faster with $1/r^2$

$$\Phi = \frac{q}{4\pi\epsilon_o r_{ret}}\ \frac{1}{(1-\vec{\beta}\cdot\hat{r}_{ret})}, \qquad \vec{A} = \frac{q}{4\pi\epsilon_o r_{ret}}\ \frac{\vec{v}}{c^2(1-\vec{\beta}\cdot\hat{r}_{ret})} \tag{6.53}$$

$$\mathsf{E} = \frac{q}{4\pi\epsilon_o r_{ret}}\ \frac{\hat{r}_{ret}\times(\vec{r}_{ph}\times\vec{a})}{c^2(1-\vec{\beta}\cdot\hat{r}_{ret})^3}, \qquad \mathsf{B} = \frac{1}{c}\ (\hat{r}_{ret}\times\mathsf{E}) \tag{6.54}$$

$$\text{Where:} \left\{ \begin{array}{lcl} r_{ret} & = & \text{distance from the retarded charge.} \\ \hat{r}_{ret} & = & \text{unit vector from retarded charge to } (\vec{x},t) \\ \vec{r}_{ph} & = & \text{the vector } (\hat{r}_{ret} - \vec{\beta}) \\ 1/(1-\vec{\beta}\cdot\hat{r}_{ret}) & = & \text{compression or 'shockwave' factor} \end{array} \right.$$

The electric and magnetic field are always transversal to the vector $\vec{r}_{ret}$ and to each other. This is not the case with the (vector)-potentials which can in general have four polarization components. We will further simplify these expressions with the assumption that the transition current moves with a non-relativistic speed. This is reasonable in general for atomic transition currents since the highest energy level for a hydrogen orbit is 13.6 eV corresponding with a speed of $\alpha c \approx c/137.036$.

The velocity associated with orbital momentum transition currents will in general be a fraction of this. The non-relativistic approximations have no more shockwave terms while all radii can be replaced by r. The exception is the potential field Ψ. This field would be zero (neutral) because the electron's field is canceled by the core's charge in the bound state. For Φ we have to maintain the first order "shockwave" term produced by the rotating transition current of the radiative electron. Using $1/(1-\epsilon) \approx 1+\epsilon$ we get the following expressions:

$$\Phi = \frac{q}{4\pi\epsilon_o c r}\ \vec{v}\cdot\hat{r}, \qquad \vec{A} = \frac{q\ \vec{v}}{4\pi\epsilon_o c^2 r} \tag{6.55}$$

$$\mathsf{E} = \frac{q}{4\pi\epsilon_o r}\ \frac{\hat{r}\times(\hat{r}\times\vec{a})}{c^2}, \qquad \mathsf{B} = -\frac{q}{4\pi\epsilon_o r}\ \frac{\hat{r}\times\vec{a}}{c^3} \tag{6.56}$$

The expression for B was further simplified using the identity $\hat{r}\times(\hat{r}\times(\hat{r}\times\vec{a})) = -\,\hat{r}\times\vec{a}$.

6.10 Spin density of electromagnetic radiation

We can now write down the expressions for the electromagnetic energy density $\mathcal{E}$, the momentum density $\vec{\mathcal{P}}$ the electromagnetic spin density $\vec{\mathcal{C}}$ and its two subcomponents $\vec{\mathcal{C}}_e$ and $\vec{\mathcal{C}}_m$ as derived for electromagnetic radiation from atomic spin 1 transitions.

Energy, Momentum and Spin density of electromagnetic radiation

$$\mathcal{E} = \frac{1}{2}(\epsilon_o E^2 + \frac{1}{\mu_o}B^2) = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{a^2}{c^4}\ |\ \hat{r}\times\hat{a}\ |^2 \tag{6.57}$$

$$\vec{\mathcal{P}} = \epsilon_o(\ \vec{E}\times\vec{B}\ ) = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{a^2}{c^5}\ |\ \hat{r}\times\hat{a}\ |^2\ \hat{r} \tag{6.58}$$

$$\vec{\mathcal{C}}_e = \epsilon_o(\ \vec{E}\times\vec{A}\ ) = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\ (\hat{r}\times(\hat{r}\times\hat{a}))\times\hat{v} \tag{6.59}$$

$$\vec{\mathcal{C}}_m = \frac{1}{\mu_o}(\ \vec{B}\Phi\ ) = -\frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\ (\hat{r}\times\hat{a})(\hat{v}\cdot\hat{r}) \tag{6.60}$$

$$\vec{\mathcal{C}} = \epsilon_o\vec{E}\times\vec{A} + \frac{1}{\mu_o}\vec{B}\Phi = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\ ((\hat{v}\times\hat{a})\cdot\hat{r})\ \hat{r} \tag{6.61}$$

All the vectors in the above expressions are unit vectors. The vector expressions for the momentum density and the spin density have been reworked with the help of the following standard vector identities:

$$\begin{aligned} A\times B &= -B\times A \\ A\times(B\times C) &= B(A\cdot C) - C(A\cdot B) \\ A\cdot(B\times C) &= B\cdot(C\times A) = C\cdot(A\times B) \end{aligned} \tag{6.62}$$

For the momentum density $\vec{\mathcal{P}}_s$ and the spin density $\vec{\mathcal{C}}$ as follows:

$$\begin{aligned} \vec{\mathcal{P}}_s &: \quad (\hat{r}\times\vec{a})\times(\hat{r}\times(\hat{r}\times\vec{a})) = \hat{r}((\hat{r}\times\vec{a})\cdot(\hat{r}\times\vec{a})) + 0 = |(\hat{r}\times\vec{a})|^2\ \hat{r} \\ \vec{\mathcal{C}}_e &: \quad -\vec{v}\times(\hat{r}\times(\hat{r}\times\vec{a})) = -\hat{r}(\hat{v}\cdot(\hat{r}\times\vec{a})) - (\hat{r}\times\vec{a})(\hat{v}\cdot\hat{r}) \\ \vec{\mathcal{C}} &: \quad -\hat{r}(\hat{v}\cdot(\hat{r}\times\vec{a})) = ((\hat{v}\times\hat{a})\cdot\hat{r})\ \hat{r} \end{aligned} \tag{6.63}$$

The spin density $\vec{\mathcal{C}}$ has no more time varying components since $(\hat{v}\times\hat{a})$ is a constant vector of length 1 and pointing in the direction of the source spin. $\vec{\mathcal{C}}$ points in the direction of propagation as required for a massless field while its length is a projection $(\cos\theta)$ of the original spin on the line of propagation.

$$\frac{|\ \vec{\mathcal{C}}\ |}{\mathcal{E}_{max}} = \frac{|\quad \text{spin density} \quad |}{|\quad \text{energy density} \quad |} = \frac{v}{a}\cos\theta = \frac{\hbar}{E_{max}}\cos\theta \quad (6.64)$$

$$\frac{|\ \vec{\mathcal{C}}\ |}{|\ \vec{\mathcal{P}}_{max}\ |} = \frac{|\quad \text{spin density} \quad |}{|\ \text{momentum density}\ |} = \frac{vc}{a}\cos\theta = \frac{\hbar}{p_{max}}\cos\theta \quad (6.65)$$

6.11 Total radiation angular momemtum density

We can now, like we did for the static electron fields, calculate the total angular momentum from spin for electromagnetic radiation. Comparing (6.28) with (6.66) we see that the dimensionless components between brackets are the same. The radiation field however decays slower (by a factor r^2). For the total angular momentum we first derive the effective current $\vec{\mathcal{P}}_s = \nabla \times \vec{C}$ from the spin-density via Stokes law. From this we can determine the effective angular momentum $\vec{L}_s = \vec{R} \times \vec{\mathcal{P}}_s$. The total angular momentum due to spin is then the spin density plus the effective angular momentum density.

$$\vec{\mathcal{C}} = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\left(\frac{xz}{r^2},\quad \frac{yz}{r^2},\quad \frac{zz}{r^2} \right) \quad (6.66)$$

$$\vec{\mathcal{P}}_s = \frac{q^2}{16\pi^2\epsilon_o r^3}\ \frac{av}{c^4}\left(-\frac{y}{r},\quad \frac{x}{r},\quad 0 \right) \quad (6.67)$$

$$\vec{\mathcal{L}}_s = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\left(-\frac{xz}{r^2},\quad -\frac{yz}{r^2},\quad \frac{x^2+y^2}{r^2} \right) \quad (6.68)$$

$$\vec{\mathcal{J}}_s = \frac{q^2}{16\pi^2\epsilon_o r^2}\ \frac{av}{c^4}\left(0,\quad 0,\quad 1 \right) \quad (6.69)$$

It follows that the direction of the total angular momentum is the same as the source spin. This is true for any type of radiation, circular, linear or elliptical polarized, as long as the source is a spin 1 transition current.

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