Chapter 1
Elementary solutions of the classical wave equation
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from my book:
Understanding Relativistic Quantum Field Theory

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Chapter 1

Elementary solutions of the classical wave equation

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1.1 The classical wave equation

The classical Electro Magnetic field is described by the classical Wave Equation. A one dimensional mechanical equivalent of this equation is depicted in the figure below. A two dimensional version would be a 2D grid of springs and masses where the masses are allowed to move vertically. The masses are well perturbed from their rest positions in the image below to show why the forces indicated by the arrows do occur. We can straightforwardly write down the mathematical expression.

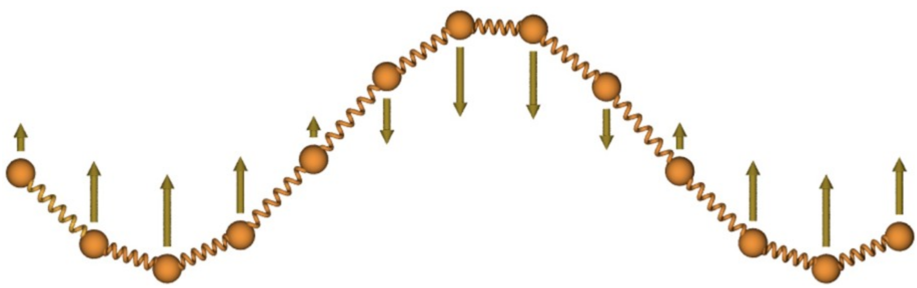

Figure 1.1: Mechanical representation of the classical wave equation

The acceleration of the masses (the second order derivative in time) is given by the force which is exerted by the springs. The force is given by the second order derivative in x, in combination with the strength of the springs given by parameter v^2:

$$\text{Classical Wave equation:} \qquad \frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2} \tag{1.1}$$

Where ψ is the vertical displacement in the mechanical model. We see that the derivatives in time and space are equal except for the parameter v^2 which determines the characteristic speed v of the medium. This simply means that the equation is satisfied by any arbitrary function which shifts along with a speed v (or $-v$). A function "stretched" by a factor v has it's slopes decreased by a factor v, while it's second order derivatives are lower by a factor v^2. When we expand this equation to three spatial dimensions we can write down the following for the electric potential field Φ and magnetic vector potential A_i

$$\text{Electric Potential:} \qquad \frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2} + c^2 \frac{\partial^2 \Phi}{\partial y^2} + c^2 \frac{\partial^2 \Phi}{\partial z^2} \tag{1.2}$$

$$\text{Mag.Vector Potential:} \qquad \frac{\partial^2 A_i}{\partial t^2} = c^2 \frac{\partial^2 A_i}{\partial x^2} + c^2 \frac{\partial^2 A_i}{\partial y^2} + c^2 \frac{\partial^2 A_i}{\partial z^2} \tag{1.3}$$

Where c is the speed of light. These are four independent equations for Φ, A_x, A_y and A_z. These equations are also satisfied by any arbitrary function which shifts along with the characteristic speed c, the electro magnetic plane waves. The plane waves can propagate in any direction. The restriction to plain waves stems from the fact that the second order spatial derivatives have to be zero in the directions orthogonal to the direction of motion.

1.2 The electron's point charge solution

We want to find stable and localized solutions of the wave equation. To do so we set the time-derivatives to zero and try to solve the remaining spatial second order equation. There are, however, no localized solutions to this equation which if we strictly require it to have zero result everywhere in space, but we can find solutions if we are slightly less restrictive and require that the result is zero everywhere except at a single point in space:

$$\nabla^2 \Phi = \delta(x)\delta(y)\delta(z), \qquad \text{with} \quad \nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \tag{1.4}$$

This delta function at $r = 0$ is now associated with the electric point charge, and the resulting potential field (normalized according to SI) we find is $\Phi = q/(4\pi\epsilon_0 r)$. In the mechanical equivalent we can the point charge as exerting a constant force on a single location of the grid. Without charge there would be no solutions which move at a speed lower as the speed of light.

$$\text{Electrostatic Potential of a point charge:} \qquad \Phi = \frac{q}{4\pi\epsilon_0 r} \tag{1.5}$$

We can see that, at every point except c(r=0), some of the second order derivatives are negative while others, in orthogonal directions, are positive.

They cancel each other to produce a zero sum. The solutions for the vector potential A, for a sufficiently slowly moving charge is also a $1/r$ field where the singularity at zero now represents a current, a moving charge. The components of A are proportional to v_x, v_y and v_z as well as proportional to the charge q itself.

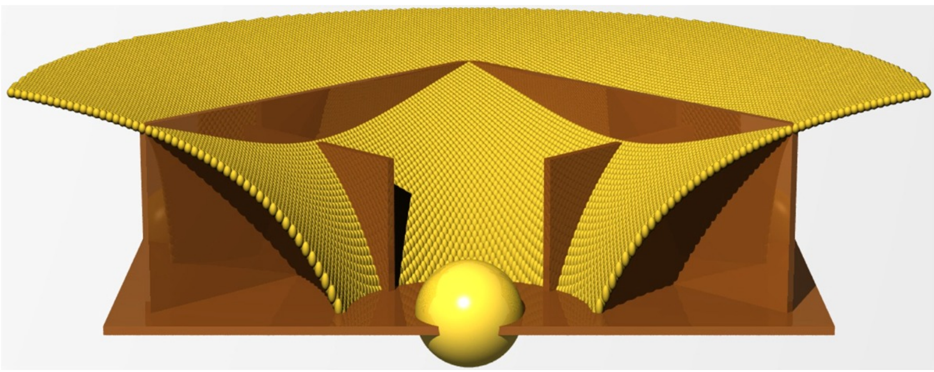

Figure 1.2: 1/r potential field of a point charge

Every linear combination of this solution is also a solution since the Wave equation is linear. We can replace the delta function with the distributed charge cloud belonging to a more realistic quantum mechanical representation of the electron. For instance the Schrödinger equation where we may consider $q\Psi^*\Psi$ as a linear sum of delta functions.

$$\nabla^2\Phi \quad = \quad q\Psi^*\Psi \tag{1.6}$$

1.3 The vector dipole solution

Other important solutions are the dipole solutions, the most important being the magnetic moment of the electron associated with it's spin. This is, as we will see, a so called axial vector. We will however, first handle the electric dipole moment, which is a vector dipole moment. The electric dipole is a combination of two adjacent delta functions (charges), one positive and on negative. We can construct a dipole by taking the spatial derivative of a delta function, and we can obtain the solution by simply taking the derivative of the solution for the delta function in the direction of the dipole, for which we will take the z-axis.

$$\nabla^2\Phi \;=\; \frac{\partial}{\partial z}\,\delta(x)\delta(y)\delta(z) \;=\; \delta(x)\delta(y)\,\frac{\partial}{\partial z}\,\delta(z) \tag{1.7}$$

Electrostatic Potential of a dipole charge: $$\Phi \;=\; \frac{\partial}{\partial z}\left(\frac{q}{r}\right) \;=\; \frac{z}{r^3} \tag{1.8}$$

A vector dipole moment is associated with a vector. A quantity which changes sign under parity inversion. (x,y,z $\Rightarrow$ -x,-y,-z) The vector is along the line which goes through the two opposite charges.

1.4 The electron's magnetic axial dipole moment

Now we want to determine the fields belonging to the electrons magnetic moment. We assume that the spin is in the z-direction. We determine the field first for a point source. Later, we will use a more realistic description where the electron is associated with a wave function which presents a charge distribution as well as a spin distribution. This is the 1934 Pauli-Weisskopf interpretation of the wave function which complements the probabilistic interpretation. This charge and spin distribution interpretation of the wave function is the foundation of a large and successful industry of numerical atomic, molecular and solid state, modeling software packages.

First we want to define a point like circular current which we use as the source of the magnetic vector potential field which represents the electron's magnetic moment. We then substitute this source in the wave equations of the magnetic vector potential. To create a point like circular current we start with a delta function, which now represents a current because we are using in the vector potential equations. We differentiate the delta function in the x and y directions for A_y and A_x, components: Two orthogonal skews yield a rotation.

Two skews represent a circular current:

$$\partial_y - \partial_x \;=\; \leftrightarrows + \downarrow\uparrow \;=\; \circlearrowleft \tag{1.9}$$

$$J_x \;=\; \frac{\partial}{\partial y}\,\delta(x,y,z), \qquad J_y \;=\; -\frac{\partial}{\partial x}\,\delta(x,y,z), \qquad J_z \;=\; 0 \tag{1.10}$$

The solutions are found (like in the electric dipole case) by differentiating the $1/r$ solution also in the x and y direction. In the solution below we use μ_o as the permeability of the vacuum and μ_e as the inherent magnetic moment of the electron.

$$\nabla^2 A_x = \frac{\mu_o\mu_e}{4\pi}\frac{\partial}{\partial y}\,\delta(x,y,z), \qquad \nabla^2 A_y = -\frac{\mu_o\mu_e}{4\pi}\frac{\partial}{\partial x}\,\delta(x,y,z) \quad (1.11)$$

$$\text{Magnetic moment:} \qquad A_x = -\frac{\mu_o\mu_e}{4\pi}\frac{y}{r^3}, \quad A_y = \frac{\mu_o\mu_e}{4\pi}\frac{x}{r^3} \quad (1.12)$$

The magnetic moment is an axial dipole moment. It is associated with two (orthogonal) vectors which span a surface, a quantity which changes sign under parity inversion. (x,y,z $\Rightarrow$ -x,-y,-z) The axial vector is associated with a circular flow through a surface. If we invert the two axis spanning the surface, then the clockwise direction of the current stays the same.

1.5 The E,B fields of the electron's charge and spin

We have now obtained the electromagnetic potential fields as stable localized solutions of the classical wave equation. For completeness we derive the fields using Maxwell's equations. In the chapter on the Interactions we will show how these equations follow from first principles.

$$\mathsf{E} = -\mathsf{grad}\ \Phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathsf{B} = \mathsf{curl}\ \mathbf{A} \quad (1.13)$$

Electron's electromagnetic fields from charge and magnetic moment:

$$\mathsf{E}_x = \frac{q}{4\pi\epsilon_o}\,\frac{x}{r^3}, \quad \mathsf{E}_y = \frac{q}{4\pi\epsilon_o}\,\frac{y}{r^3}, \quad \mathsf{E}_z = \frac{q}{4\pi\epsilon_o}\,\frac{z}{r^3} \quad (1.14)$$

$$\mathsf{B}_x = \frac{\mu_o\mu_e}{4\pi}\left(\frac{3z}{r^5}x\right), \quad \mathsf{B}_y = \frac{\mu_o\mu_e}{4\pi}\left(\frac{3z}{r^5}y\right), \quad \mathsf{B}_z = \frac{\mu_o\mu_e}{4\pi}\left(\frac{3z}{r^5}z - \frac{1}{r^3}\right) \quad (1.15)$$

The Electric field E is a (true) vector field, also called a 1-form where the comes from 1 dimension. The magnetic field B is a pseudo (= axial) vector. Axial vectors are associated with a surface spanned by two vectors (2-form). It would have been more correct to give the magnetic field component dual indices like B_{xy} instead of B_z since this habit of using the remaining third dimension as an index is a trick which only works in 3d and not in 2d, 4d or higher dimensions.

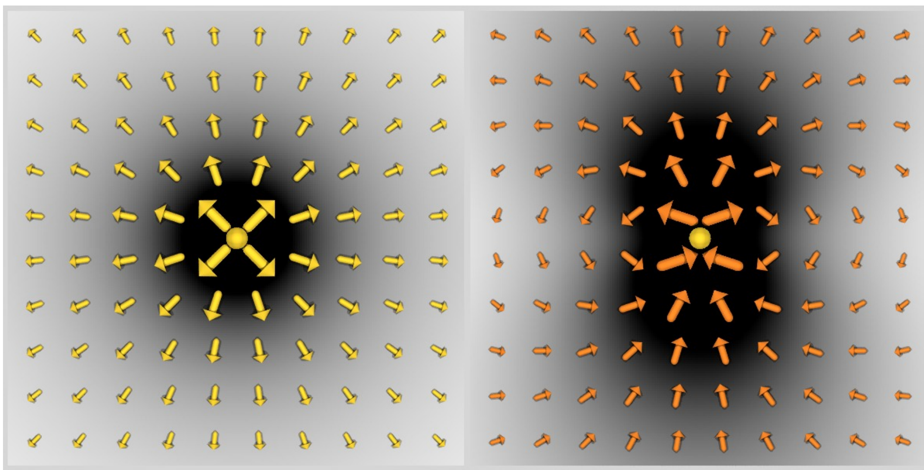

Figure 1.3: Electric and magnetic fields of the electron

Anyway, it is important to realize the true nature of axial vectors versus true vectors. Always when we talk about the "*direction*" of spin, magnetic fields and dipole moments we should have in the back of our minds that we are in fact misusing the proper meaning of direction.

1.6 Magnetic monopoles

It is interesting to show that if there would exist magnetic monopoles then we could derive the field of the electron's magnetic moment in the same way as we derived the electric dipole moment, that is by taking the derivative of the delta function in the z-direction creating a positive and negative magnetic charge next to each other.

$$\mathbf{B} \;=\; -\mathsf{grad}\;(\;\Phi_m\;)\;=\;\mathsf{curl}\;(\;A\;) \tag{1.16}$$

$$\mathbf{B} \;=\; -\frac{\mu_o\mu_e}{4\pi}\;\mathsf{grad}\left(\,\frac{z}{r^3}\right) \;=\; \frac{\mu_o\mu_e}{4\pi}\;\mathsf{curl}\left(\,-\frac{y}{r^3},\;\frac{x}{r^3},0\,\right) \tag{1.17}$$

We would have an extra set of potential fields V_m and $\mathbf{A}_m$ associated with magnetic charges which are dual to the ones of the electric charge. Taking the gradient of such a magnetic scalar potential V_m gives exactly same results for the B fields as the curl of the standard A vector potential does. This shows that we indeed may call the magnetic moment a dipole moment, even though the potential fields V_m and $\mathbf{A}_m$ and the associated magnetic charge do not exist, or at least, never have been observed.

1.7 Physical limitations of point charge and spin

For sofar we have derived the solutions of the Classical Wave equation corresponding to the potential fields of the point like electron, due to its electric point charge and magnetic moment. Any linear combination of this solution is also a solution of the Wave equation. Classically there are a number of problems with fields from point like sources. One of the more severe is that energy and momenta and spin of these fields become infinite if we do not apply some sort of cut-off at small distances. For the energy of the electro static and magneto static fields we calculate for a cut-off radius of r_o

$$\text{Energy density in } J/m^3: \qquad U \;=\; \frac{1}{2}\left(\epsilon_o\mathsf{E}^2 + \frac{1}{\mu_o}\mathsf{B}^2\right) \tag{1.18}$$

$$\text{Electric energy:} \qquad \frac{q^2}{8\pi\epsilon_o r_o} \;=\; \int_{r_o}^{\infty} dr^3\;4\pi r^2\;\frac{q^2}{32\pi^2\epsilon_o r^4} \tag{1.19}$$

$$\text{Magnetic energy:} \quad \frac{\mu_o\mu_e^2}{12\pi r_o^3} \;=\; \int_{r_o}^{\infty} dr^3\;4\pi r^2\;\frac{\mu_o\mu_e^2}{32\pi^2}\left(\frac{3z^2}{r^8}+\frac{1}{r^6}\right) \tag{1.20}$$

Where the first result term corresponds to the electro static field energy. If the cut-off radius r_o becomes smaller as half the classical electron radius then the energy of the field becomes larger as the rest mass of the electron: $r_o = r_e/2 = 1.4089701625\;10^{-15}\;m$.

The rest mass energy is reached sooner, at $r_o = 3.27413591\;10^{-14}\;m$, if we consider the magnetostatic energy associated with the magnetic moment.

Another limit is the inherent spin of the electron. We can use the Poynting vector which represents the energy-flux, in Joule going through one unit of area during one unit of time, to obtain the effective momentum density $\mathcal{P}$ expressed in Js/m^4, that is, momentum Js/m per unit of volume $1/m^3$.

$$\text{Poynting vector in } J/(m^2s): \qquad = \frac{1}{\mu_o}\,\mathsf{E}\times\mathsf{B} \tag{1.21}$$

$$\mathcal{P}_x = -\frac{q\mu_e}{16\pi^2\epsilon_o c^2}\,\frac{y}{r^6}, \qquad \mathcal{P}_y = \frac{q\mu_e}{16\pi^2\epsilon_o c^2}\,\frac{x}{r^6}, \qquad \mathcal{P}_z = 0 \tag{1.22}$$

Interpreting this as the z-component of the effective angular momentum density, $\vec{\mathcal{S}_z} = \vec{r}\times\vec{\mathcal{P}}$, expressed in Js/m^3, which, after integration, gives us the effective angular moment component s_z contribution from the EM field given in Joule seconds.

$$\text{Angular mom. density} \quad = \frac{q\mu_e}{16\pi^2\epsilon_o c^2}\left\{\frac{-xz}{r^6},\ \frac{-yz}{r^6},\ \frac{x^2+y^2}{r^6}\right\} \tag{1.23}$$

$$s_z = \frac{q\mu_e}{6\pi\epsilon_o c^2 r_o} = \int_{r_0}^{\infty} dr\ r^2 \int_0^{\pi} d\theta\ 2\pi\sin\theta\left(\frac{q\mu_e}{16\pi^2\epsilon_o c^2}\,\frac{\sin^2\theta}{r^4}\right) \tag{1.24}$$

The spin of the electron is, like for any fermion, $\hbar/2$. If we now calculate the cut-off radius at which the spin becomes $\hbar/2$ we find $r_o =$ 1.8808053359 10^{-15} m, which is 2/3 of the classical electron radius. (times 1.00115965218085 which corresponds to the anomalous magnetic moment correction of the electron.)

1.8 The Pauli-Weisskopf interpretation

All these quantities become infinite if r_o goes to zero which leads us to the limits of the classical image of the electron as a point particle. This is where the Pauli-Weisskopf interpretation of the wave function as a charge, current density function comes into play. This interpretation complements the probability interpretation and was originally proposed to explain the negative probabilities possible in the relativistic Quantum mechanical equations in 1934 by Wolfgang Pauli and Victor F. Weisskopf. The negative probability solutions are interpreted to be anti-particle solutions with opposite charge. Spreading the point-charge out over space into a charge distribution eliminates the infinities. Each point in space within the wave function is still considered to be an, infinitely small, point charge, however the total field energy stay finite. The reason is the dependence of the energy on the square of the fields: Separating a single point charge into two separate point charges halves the total energy. Probability densities ($1/m^3$) and charge densities (C/m^3) are accompanied by probability fluxes, $1/(m^2s)$ and charge current densities, $C/(m^2s)$, to form four vectors which can be properly Lorentz transformed.

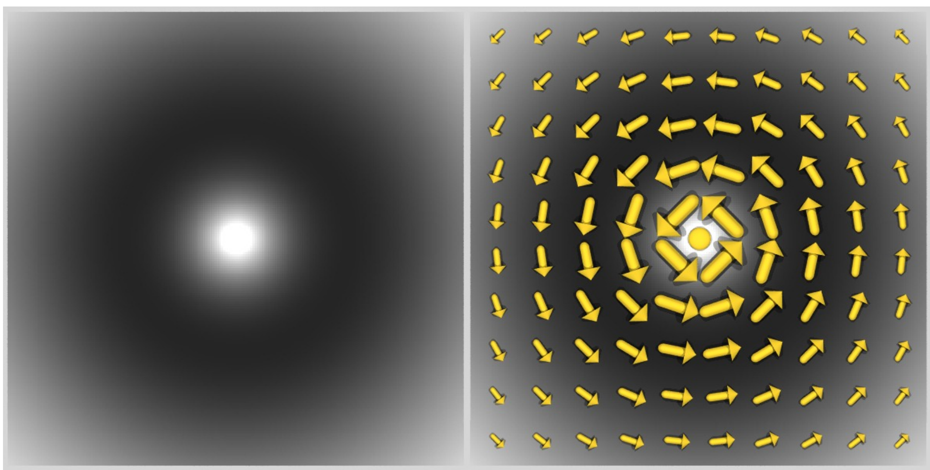

Figure 1.4: Left: charge density, right: angular mom. current density

The image above, figure (1.4) represents the charge-current density of a hydrogen 2P-state state with angular momentum one. A continuous charge-current density like this does not radiate electro magnetically because it represents stationary situation, even though there are currents. Each point of the wave function produces stationary V and A potential fields which, by definition, do not radiate. Therefor, atomic electron orbits do not radiate per definition in the Pauli-Weisskopf interpretation.

There is a vast base of experimental and industrial verification for the Pauli-Weisskopf interpretation. Indeed, in molecular modeling one has to assume a continuously distributed charge-current and spin density in order to correctly predict the properties of atoms, molecules and solid state materials. Widely applied techniques like Density Functional Theory and more recently Quantum Molecular Dynamics, which includes the motion of the nuclei have shown to be capable of ever better predicting and simulating the properties of materials, from elasticity and stress-strain relations, to heat capacity, the complex phase transitions under pressure and temperature, the conductive properties, metallic, isolator, semiconductor, fermi levels, bands and band-gaps, photo emission and absorbtion, the magnetic properties, ferroelectric properties, electron-phonon interactions, interface properties between different materials, et-cetera.

1.9 Gordon decomposition and spin current

As we will see later, when the Dirac theory of the electron is discussed, the wave function for spin 1/2 fermions automatically includes a spin term and this inherent term gives rise to an effective charge-current density equal to the curl of the spin-density. That is, the wave function also represents a spin density where spin is to be understood in the sense of the point spin as discussed above but distributed continuously over the wave function. This effect is very similar to that of a classical magnetic material which exhibits an *effective* current equal to the curl of the magnetic dipole density:

$$j_{eff} \;=\; \nabla\times\mathsf{M} \tag{1.25}$$

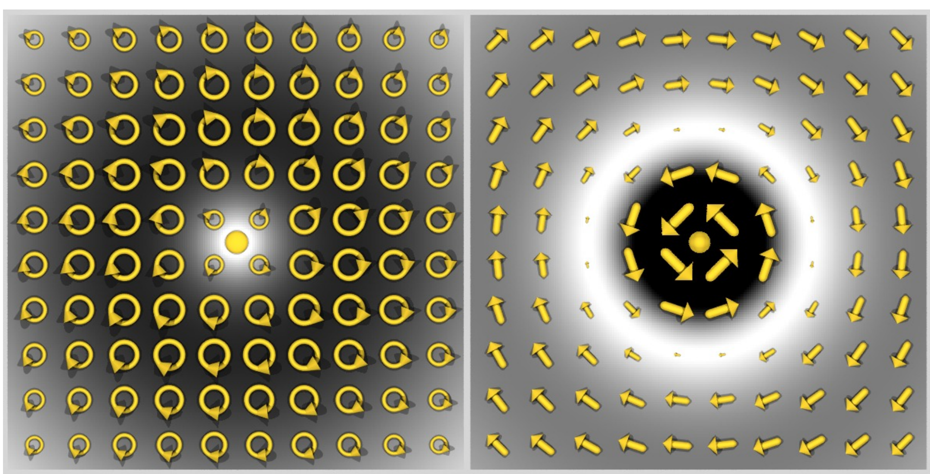

Figure 1.5: Left: spin density, right: effective spin current density

Now, recall Stokes law: Inside the magnetic material the little circular currents cancel each other if the spin density is constant. However they don't cancel at the edge or where there is a gradient. This then gives rise to a large *effective* current surrounding the magnetic material at the edge. This is the same which happens with the electron's wave function, both electrically in the form of an effective current surrounding the wave function, as well as inertially in the form of an effective momentum flow around the wave function.

Figure (1.5) depicts the spin-density at the left side for a 2P hydrogen state, but now with an inherent spin $s_z = \hbar/2$, in a direction counter wise to that of the angular momentum shown in fig. 1.4 The image at the right is the equivalent of the image on the left. Note how the effective current, at the right, becomes zero in those areas where the spin density is constant, and how it becomes larger in the areas where there is a gradient in the spin density. The outer areas show the effective spin current as being clockwise like the spin component itself, however, in the inner region the effective current is contrary to the actual spin itself because of the gradient is opposite in this area. It shows that the effective spin current is a "differential" effect.

Note that we have never explicitly tried to use the charge, responsible for the electric field, as the rotating source of the magnetic moment of the electron. We won't do so here because it's simply not possible. The problem is that the charge would have to move at a speed higher then the light speed if rotates in a circle with a radius less than the Compton radius of the electron ($3.86159\ 10^{-13} m$) according to the law for the magnetic moment:

$$\mu \;=\; IA, \qquad I = \frac{c}{2\pi r_c} q, \qquad A = \pi r_c^2 \tag{1.26}$$

Where r_c is the Compton radius, I is the current and A is the total area enclosed by the current. Now, the Compton radius is much to large to be seriously considered, but there are however other options available in a vacuum which supports virtual particle pairs with opposite charge. For example, consider the case of a virtual particle pair with opposite spin and charge. The spins and charges cancel but the magnetic moments add to produce a non-zero magnetic vector potential. We bring up this example just to shows how the vacuum could support a magnetic spin density without the principle limitations of the Compton radius and without necessary being connected one-to-one with the motion of the charge density.

1.10 Plane wave interference

If a free electron suddenly changes course in a scattering experiment then we can describe this process to a good degree of accuracy by a superposition of two plane waves. One for the *initial* state of the electron and one for the *final* state of the electron.

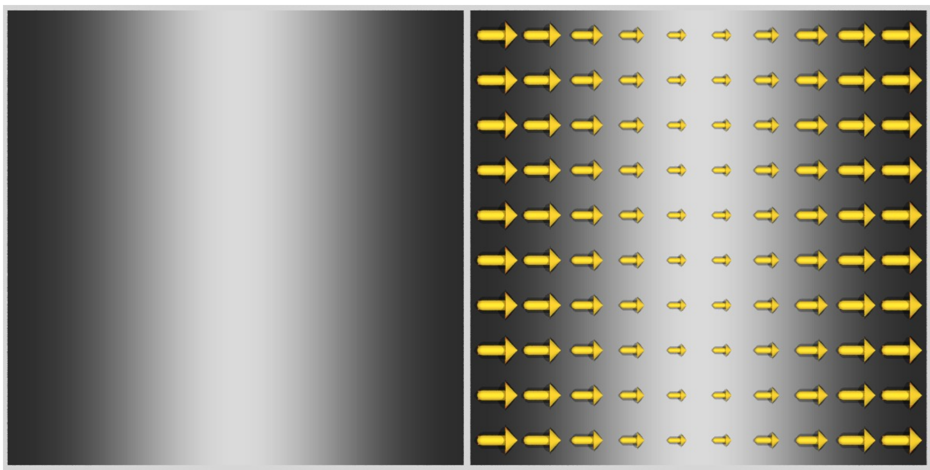

Figure 1.6: A zoom-in on plane wave charge and current density

When we calculate the charge and current densities we find interference terms which constitute a third plane wave. The frequency and wavelength of this plane wave obey the laws of energy/momentum conservation.

$$(E_1, \vec{p}_1) - (E_2, \vec{p}_2) \;=\; (E_3, \vec{p}_3) \tag{1.27}$$

The interference term is real in the form of a sinusoidal wave. In figure 1.6 we zoom into the plane wave, the image shows the charge and current density. We see a charge distribution shifting alternatingly in the direction of the propagation, which does not provide the transversely alternating currents associated with electromagnetic radiation.

However if we look at the spin density in figure 1.7 then we see that the effective current resulting from a changing spin density ($J_s = \mathsf{curl}\ \vec{\mathcal{S}}$) *does* provide an transversely alternating current which acts as the source of transversal electromagnetic radiation. We see that the intrinsic spin plays an essential role in the absorbtion and emission of (virtual) photons.

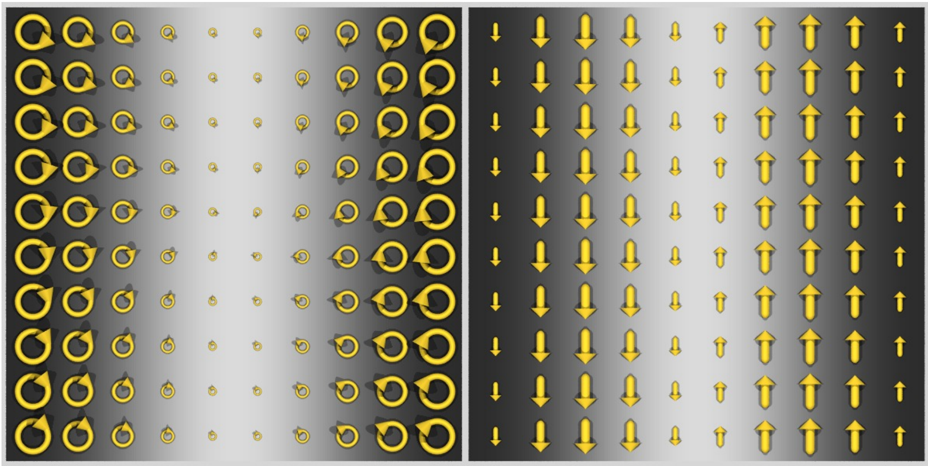

Figure 1.7: Plane wave spin density and effective spin current density

1.11 Propagation on the light cone

In section 1.2 we derived the elementary solutions of the wave equation, specifically the $1/r$ potential belonging to the point source $\delta(r)$. Here we want to discuss an even more elementary solution: The response to an instantaneous impulse $\delta(t)\delta(r)$. This impulse response is known as the *Green's function* of the field or the *propagator*. This delta function in both space and time can be considered as an elementary building block from which any arbitrary function can be constructed due to the linear character of the wave equation.

We will not only discuss the propagator for the usual $3d$ space but for any dimension to show that our three dimensional world with a time dimension represents a very special case in which the form of the (photon) propagator is far more elegant as it is in other dimensions. Looking at fig. 1.8, where the 3d propagation is shown in the middle of higher and lower dimensional spaces, we see that the delta pulse is propagated, on the light cone, as a delta pulse as well. Any event reaches, and influences, other events just once at a time determined by the speed of light c.

The propagators in higher dimensional spaces are differentials of the 3d propagator. The 5d propagator is the first order differential of the Dirac pulse, the 7d propagator is the second, and so on. The propagators for even dimensional spaces are fractional order differentials ($4d \rightarrow 1/2,\ \ 6d \rightarrow 3/2, ...$). The important 1d propagator which describes plane wave propagation is the integral of the 3d propagator, while the 2d propagator which applies to surfaces is the half integral of the 3d-propagator.

The propagators in the drawing were obtained by computer lattice simulations where the delta functions were replaced by a narrow Gaussian function.

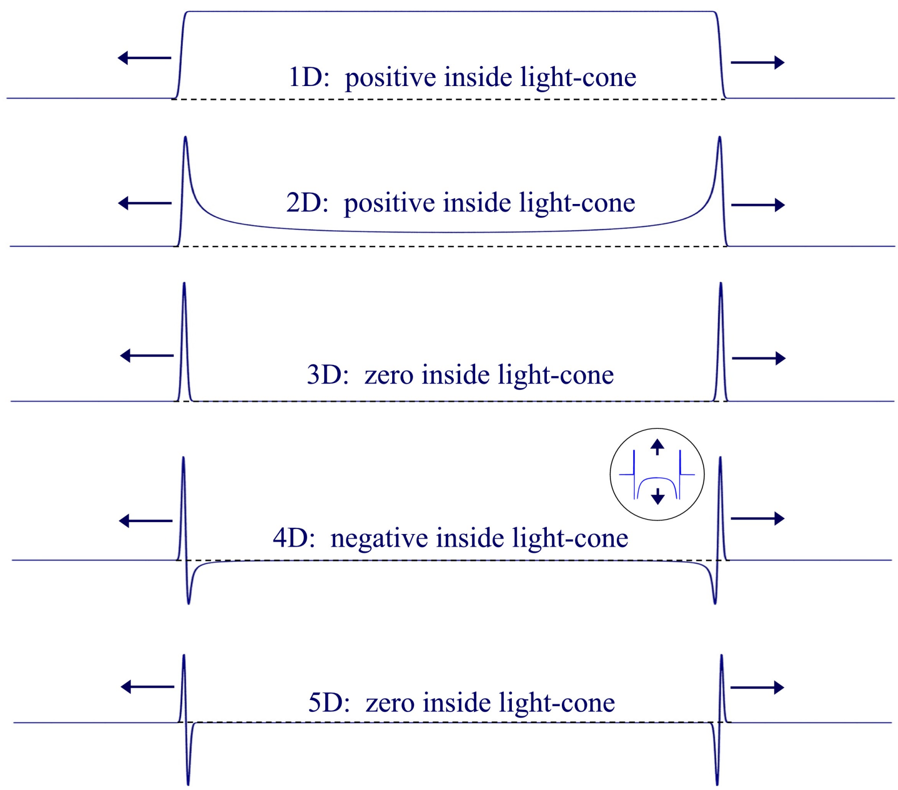

Figure 1.8: d-dimensional propagators, radiating away on the r-axis

We will also pay significant attention to the $1d$ propagator since it represents the behavior of plane waves in general, plane waves are basically one dimensional objects in any dimension, that is, they vary only in one dimension. Plane waves can be described by a single energy and momentum, they are the momentum eigen-states of the wave equation. The Fourier transform, in any dimension, is a decomposition into plane waves. The Fourier transformed space becomes the momentum -space and calculations done in this space can be done algebraically with the eigen-values instead of differential operators.

Propagators in momentum space have always the same form regardless of the number of dimensions because the eigen functions, the plane waves, are in principle always one dimensional objects. Extrapolating a propagator to a higher or lower dimension just amounts to adapting the n-dimensional momentum $p^2 = p_{x1}^2 + p_{x2}^2 + ...p_{xn}^2$. For these reasons we will first derive the propagator in momentum space starting with the wave function in *configuration space* (Which is just the usual time-space representation). The equation to solve now does include the time-derivatives in contrast to the equation for the stable solutions. We can write the same expression in momentum space by replacing the derivatives with the eigenvalues:

$$+i\hbar\frac{\partial}{\partial t}\ \left(e^{-iEt/\hbar+ipx/\hbar}\right) \;=\; E\left(e^{-iEt/\hbar+ipx/\hbar}\right) \tag{1.28}$$

$$-i\hbar\frac{\partial}{\partial x}\ \left(e^{-iEt/\hbar+ipx/\hbar}\right) \;=\; p\left(e^{-iEt/\hbar+ipx/\hbar}\right) \tag{1.29}$$

Since the Fourier transform of a Dirac function $\delta(x_1)$ is simply a constant 1 in the corresponding p_1 dimension in momentum space, and since the Fourier transform of a multiplication is a convolution, we can simply express the product of four Dirac functions $\delta(t, x, y, z)$ as a 1 in momentum space: A constant over all time and space. The Fourier transform of our equation now becomes.

configuration space:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} - c^2\frac{\partial^2}{\partial y^2} - c^2\frac{\partial^2}{\partial z^2}\right)\ \mathcal{D}(t,r) \;=\; \delta(t)\delta(x)\delta(y)\delta(z) \tag{1.30}$$

momentum space:

$$\left(E^2\ -\ c^2p_x^2\ -\ c^2p_y^2\ -\ c^2p_z^2\right)\ \mathcal{D}(E,p) \;=\; -1 \tag{1.31}$$

Where $\mathcal{D}_x$ is the propagator or Green's function in configuration space while $\mathcal{G}_p$ is the Green's function in momentum space. In momentum space the photon propagator simply becomes algebraically:

$$\mathcal{D}(E,p) \quad = \quad \frac{-1}{E^2 - c^2p_x^2 + c^2p_y^2 + c^2p_z^2} \quad = \quad -\frac{1}{q^2} \tag{1.32}$$

We have used q here in the last expression in a way which is entirely independent of the number of dimensions. Now we proceed first with the derivation of the one dimensional space propagator via the (inverse) Fourier transform $\mathcal{F}_i\mathcal{D}(E,p) = \mathcal{D}(t,r)$. We split the second order expression in first order poles and we set c=1 for simplicity.

$$\frac{-1}{E^2 - p^2} \quad = \quad \frac{1}{2p}\left(\frac{-1}{E-p} \; + \; \frac{1}{E+p}\right) \tag{1.33}$$

The Fourier transforms of the individual factors are standard table items, for instance for the transforms from momentum p to position r we have.

$$\mathcal{F}\left\{\frac{1}{p}\right\} \quad = \quad \frac{i}{2}\,\mathrm{sgn}(r) \tag{1.34}$$

The Fourier transform for a constant value 1 from the Energy to the time-domain we have:

$$\mathcal{F}\left\{1\right\} \quad = \quad \delta(t) \tag{1.35}$$

So that we can write for the 2d Fourier transform (energy/ momentum to time/space)

$$\mathcal{F}\left\{\frac{1}{2p}\right\} \quad = \quad \frac{i}{4}\,\delta(t)\;\mathrm{sgn}(r) \tag{1.36}$$

The other terms are simply 45 and 135 degrees rotated versions of the above, for instance:

$$\mathcal{F}\left\{\frac{1}{E-p}\right\} \quad = \quad \frac{i}{2}\,\delta(t+r)\;\mathrm{sgn}(t-r) \tag{1.37}$$

$$\mathcal{F}\left\{\frac{1}{E+p}\right\} \quad = \quad \frac{i}{2}\,\delta(t-r)\;\mathrm{sgn}(t+r) \tag{1.38}$$

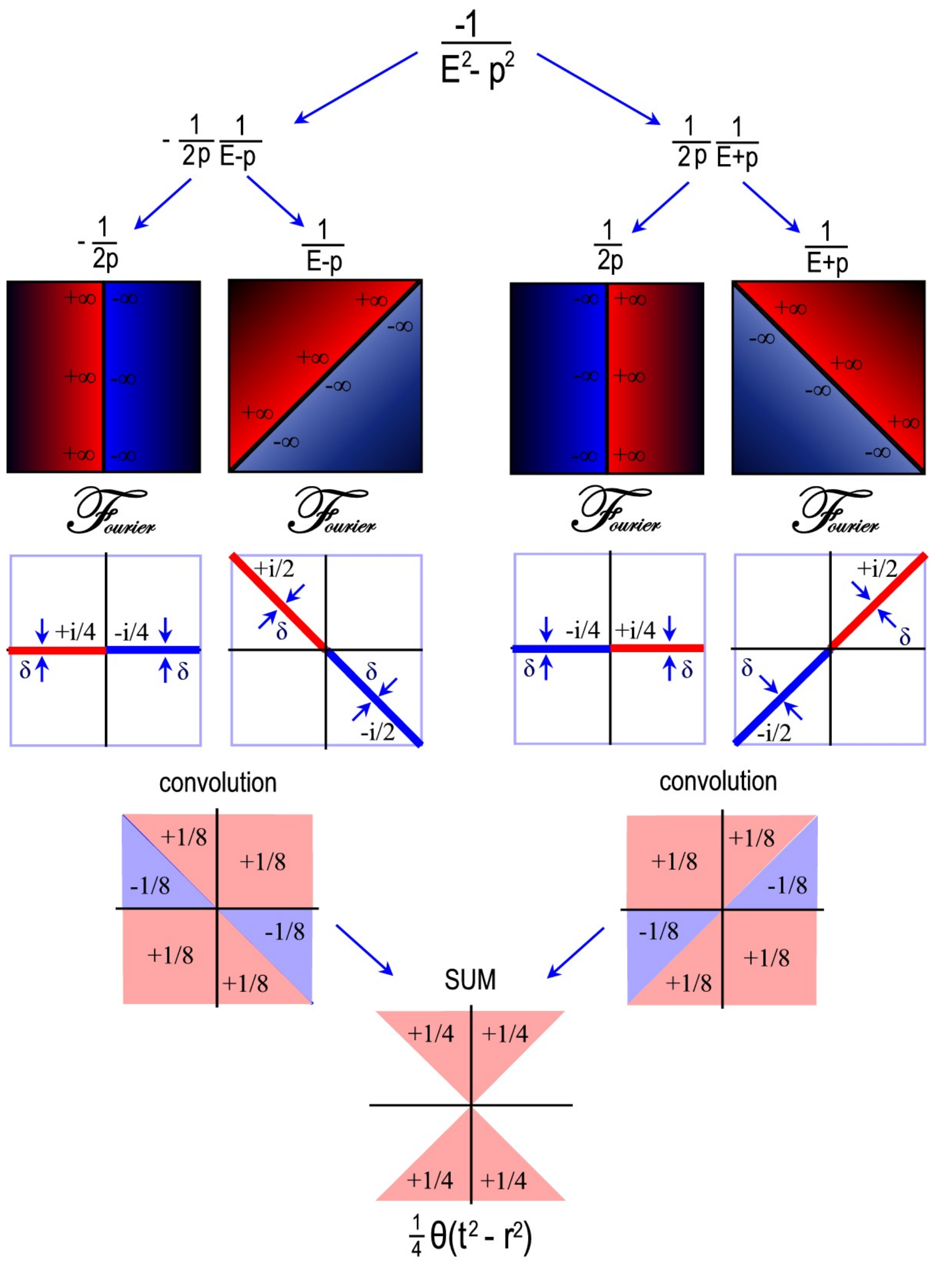

Figure 1.9: Bi-directional 1+1d Photon propagator

See figure (1.9) which shows the individual terms and their Fourier transforms. Multiplications correspond to convolutions in the Fourier domain. The convolutions result in bi-valued functions which are either $+1/8$ or $-1/8$ depending on the sub-quadrant. The final result, the Fourier transform of $-1/(E^2 - p^2)$ is shown at the bottom of figure (1.9). It is zero outside the light-cone and constant within both the forward and backward light-cone. $\mathcal{D}(t,r) = \theta(t^2 - r^2)/4$. Where θ is the Heaviside step function.

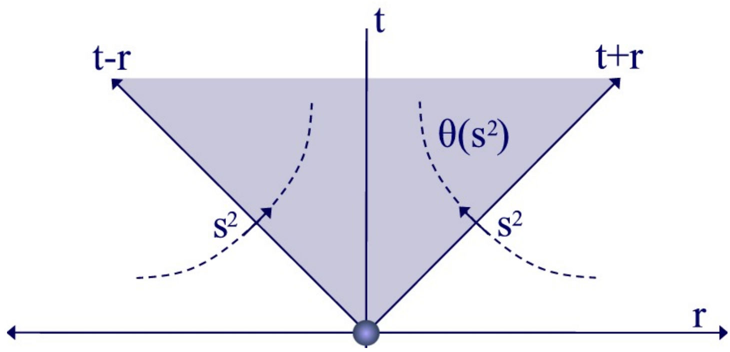

Figure 1.10: The 1+1d propagator in position space

We see that this expression propagates both forward and backward in time. We require however that the with the remark that there is no propagation for $t < 0$. Further on we will show that this requires a modification of the poles of the expression $-1/E^2 - p^2)$. In the literature this modification and its Fourier transform are generally written as :

$$\mathcal{F}\left\{\frac{-1}{(E+i\epsilon)^2 - p^2}\right\} \;=\; \mathcal{D}_1(t,r) \;=\; \frac{1}{2}\,\theta(t)\,\theta(s^2) \tag{1.39}$$

Where ϵ is an infinitely small positive constant and $s^2 = t^2 - r^2$. The Fourier transform of this 2d propagator is shown in figure (1.10). It is only non-zero at $(t > 0)$. This is the plane wave propagator. We will go further with this expression and later study all the aspects of the above ϵ-prescription.

The Green's function shown in figure (1.10) is a rectangle with constant amplitude which spreads with the speed of light as shown in fig.1.8. ($s^2 = 0$ on the light-cone). Using this propagator as a starting point we can derive the higher dimensional photon propagators with a trick. We use an *inter dimensional* operator to derive all the higher dimensional photon propagators in configuration space directly from our 1d photon propagator result. This *inter-dimensional* operator works on any radially symmetric propagator.

$$\mathcal{D}_d(t,r) \;=\; \frac{1}{\pi^a}\,\frac{\partial^a}{\partial(s^2)^a}\,\mathcal{D}_1, \qquad (a = \tfrac{d-1}{2}) \tag{1.40}$$

We will prove this operator in an appendix, but we will discuss it to some extend further on in this section because it nicely demonstrates how the 1d (plane-wave) and the 3d (general) propagators are physically related. Applying the inter-dimensional operator on our result (eq: 1.39) we obtain:

$$\mathcal{D}_d(t,r) \;=\; \frac{\theta(t)}{2\pi^a}\,\frac{\partial^a\ \theta(s^2)}{\partial\ (s^2)^a} \tag{1.41}$$

This formula is especially simple for dimensions for which the value a is integer, that is, for 3d, 5d, et-cetera. The propagators for these dimensions are the derivatives of the spreading Heaviside step function. For 3d we thus get the Dirac function propagating on the light-cone as we did see in fig. 1.8. The 5d propagator is also propagating on the light-cone but is know the derivative of a delta function, even though the originating source is simply a delta-function. Written out we get for the photon propagators in 3d and 5d configuration space:

$$\text{3d photon propagator:}\quad \mathcal{D}_3(t,r) \;=\; \frac{\theta(t)}{2\pi}\,\delta(s^2) \tag{1.42}$$

$$\text{5d photon propagator:}\quad \mathcal{D}_5(t,r) \;=\; \frac{\theta(t)}{2\pi^2}\,\frac{\partial\ \delta(s^2)}{\partial\ (s^2)} \tag{1.43}$$

This Dirac function $\delta(s^2)$ becomes smaller overtime (distance traveled) by a factor $|\partial_r s^2| = 2r$, since the "volume"of the delta function is given by: one over the absolute value of the derivative of it's argument. We can write the 3d propagator as.

$$\mathcal{D}_3(t,r) \;=\; \frac{\theta(t)}{4\pi r}\,\delta(t-|r|) \tag{1.44}$$

Where $\delta(t-|r|)$ is also a sphere expanding on the light cone but with constant amplitude. We now use the propagator as an operator on the trajectory of an electron at rest. The propagator is an expanding sphere on the light cone decreasing with $1/r$. The electron is the line r=0 with a constant amplitude. If we integrate the propagated field which originated from the electron source from t is minus infinity to zero then we see we obtain the potential field (equation 1.5) of a stationary source.

$$\Phi \;=\; \frac{q}{\epsilon_o}\int_{-\infty}^{0} dt\; \mathcal{D}_3(t,r) \;=\; \frac{q}{\epsilon_o}\int_{-\infty}^{0} dt\; \frac{\theta(t)}{4\pi r}\;\delta(t-r) \quad\Rightarrow \tag{1.45}$$

$$\Phi \;=\; \frac{q}{4\pi\epsilon_o r} \tag{1.46}$$

The propagator can be used to determine the potential field of any electron trajectory where it doesn't matter if the speed of the electron is constant or not. This method was first developed by Liènard and Wiechert around 1900 and the resulting potentials are called the Liènard-Wiechert potentials.

1.12 Discussion of the inter dimensional operator

We did use the inter dimensional operator (eq:1.40 to derive the 3d propagator from the 1d propagator. The 1d operator in 3d space represents the response from the EM field on a delta function $\delta(t)\delta(x)$ which represents an infinite plane spanned on the y and z-axis which is given a uniform charge at t=0. The 1d propagator is therefor useful to calculate plane wave propagation. We should obtain the 1d operator if we use the 3d propagator explicitly to calculate the response on $\delta(t)\delta(x)$ by integrating over the plane. We will do so to check our result but also as an illustration to show how the propagators of different dimensions interrelate. The method we use is general valid for any propagator, as well as for going from any d-dimensional propagator to the (d-2) dimensional propagator.
The integral should obtain the total value of all contributions from the plane. These contributions will first come from the closest point on the plane and then from an ever increasing circle. At any specific time there will be an R_{max} for this circle. We get for the general formula:

$$\mathcal{D}_1(t,r) \;=\; 2\pi\int_0^{R_{max}} dR\;\; \mathcal{D}_3\left(t, \sqrt{r^2+R^2}\right) R \tag{1.47}$$

$$r = x, \qquad R = \sqrt{y^2+z^2}, \qquad s^2 = t^2 - x^2, \qquad S^2 = t^2 - x^2 - y^2 - z^2 \tag{1.48}$$

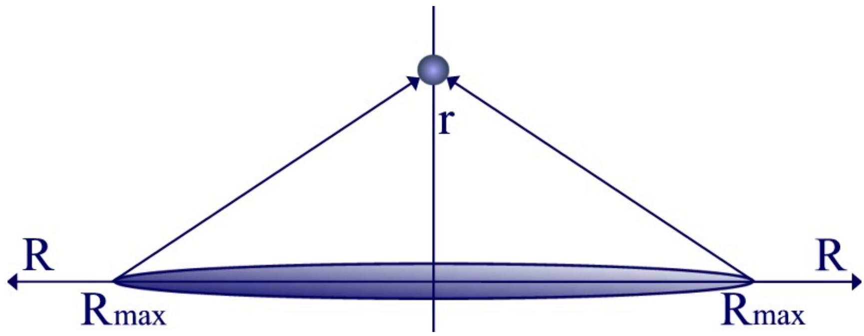

Figure 1.11: Going from 3d to 1d propagator

where (1.48) shows the conditions when we go specifically from 3d to 1d. We now proceed with this formula, we see that we can use s^2 and S^2 as the arguments of the propagator. We change the integrating variable to R^2 instead of R. Next we limit the propagation speed to smaller or equal to the light speed so we get an explicit expression for the maximum distance R_{max}. We obtain:

$$\mathcal{D}_1(s^2) = \pi \int_0^{s^2} d(R^2) \;\; \mathcal{D}_3(S^2) \tag{1.49}$$

$$\text{using:} \quad d(R^2) = 2R\; dR, \quad R^2_{max} = t^2 - r^2 = s^2 \tag{1.50}$$

Finally we replace the integration variable R^2 with S^2 using $\mathrm{d}R^2 = d(s^2 - S^2) = -dS^2$. This changes the sign of the integral but it also swaps the values of the boundaries. Swapping the boundaries back then also reverses the sign back.

$$\mathcal{D}_1(s^2) = \pi \int_0^{s^2} d(S^2) \;\; \mathcal{D}_3\left(S^2\right) \tag{1.51}$$

This is exactly the reverse operation of the one we used when we derived the 3d propagator from the 1d propagator. This is what we wanted to show. Expression (1.51) represents the inter-dimensional operator to derive the d-dimensional propagator from the (d+2) dimensional propagator. The general prove of the inter-dimensional operator is given in an appendix.

1.13 Causal propagators and the Hilbert Transform

A strictly causal propagator for the Electromagnetic potentials should propagate only within the forward light-cone. However, the propagator $1/(E^2 - p^2)$ is symmetrical in E and thus also symmetrical in time according to the symmetry-properties of the Fourier transform.

The propagator is *Even* in time, $\mathcal{D}(-t,r) = \mathcal{D}(t,r)$. We need to find the so-called *Hilbert partner* of the momentum-space propagator which represents the *Odd* propagator $\mathcal{D}(-t,r) = -\ \mathcal{D}(t,r)$, so that we can write:

$\frac{1}{2}(Even + Odd)$ = Forward propagator.

$\frac{1}{2}(Even - Odd)$ = Backward propagator.

The antisymmetric Odd propagator is simply the symmetric propagator multiplied by the sign function which we preferably define as in the scientific floating point way using +0 and -0 where $1/(-0) = -\infty$

$$\mathrm{sgn}(t) \;=\; \begin{cases} +1 \;\text{ for } & +0 \;\leq\; t \;\leq\; +\infty \\ -1 \;\text{ for } & -0 \;\geq\; t \;\geq\; -\infty \end{cases} \tag{1.52}$$

So that sgn(t) is nowhere 0. For momentum space we have the Fourier transforms.

$$\mathcal{F}\{\ 1\ \} \;=\; \delta(E), \qquad \mathcal{F}\{\ \mathrm{sgn}(t)\ \} \;=\; -\frac{i}{\pi E} \tag{1.53}$$

The forward propagator in momentum space stays unchanged under convolution with the Fourier transform of the Heaviside step function.

$$\mathcal{F}\{\ \theta(t)\ \} \;=\; \frac{1}{2}\left(\delta(E) - \frac{i}{\pi E}\right) \tag{1.54}$$

From $\mathrm{sgn}^2(t) = 1$ we see that the auto convolution of its Fourier transform gives.

$$\left(-\frac{i}{\pi E}\right) * \left(-\frac{i}{\pi E}\right) \;=\; \delta(E) \tag{1.55}$$

We can use this to check the auto-convolution of the Fourier transform of the Heaviside step function, since $\theta^2(t) = \theta(t)$ we have.

$$\frac{1}{2}\left(\delta(E) - \frac{i}{\pi E}\right) * \frac{1}{2}\left(\delta(E) - \frac{i}{\pi E}\right) \;=\; \frac{1}{2}\left(\delta(E) - \frac{i}{\pi E}\right) \qquad (1.56)$$

For an expression with a pole shifted away from zero we get.

$$\left(-\frac{i}{\pi E}\right) * \left(-\frac{i}{\pi(E-p)}\right) \;=\; \delta(E-p) \qquad (1.57)$$

At this stage we know enough to determine the anti-symmetric "*Hilbert partners*" of the symmetric photon and Klein Gordon propagators $1/(E^2 - p^2)$ and $1/(E^2 - p^2 - m^2)$. The convolution with $1/\pi E$ is a standard transformation known as the Hilbert transform. The Even and Odd functions are said to be a "*Hilbert pair*"

$$\left(-\frac{i}{\pi E}\right) * f(E) \;=\; -i\,\mathcal{H}\{\; f(E)\; \} \qquad (1.58)$$

1.14 The causal (forward) photon propagator

Using a superscript triangle and a subscript triangle to denote the forward and backward light-cones respectively, the symmetric in time photon propagator is given by.

$$\mathcal{D}^{+\triangledown}_{+\vartriangle}(t,r) \;=\; \frac{1}{2\pi}\,\delta(t^2 - r^2) \;=\; \frac{1}{4\pi r}\,\big(\;\delta(t-r) + \delta(t+r)\;\big) \qquad (1.59)$$

The rightmost expression is divided by an extra factor of $2r$ which stems from the first order derivative of the argument $t^2 - r^2$ which determines how fast it goes through zero and thus the height of the Dirac pulse since the latter is defined by $height \times width = constant$

The term $\delta(t+r)$ corresponds to the negative light-cone which would cause never experimentally detected "*advanced potentials*" next to the (retarded) electromagnetic potentials. The physically correct propagator is non-zero only on the forward light-cone. The Odd propagator is given by a multiplication with the sign function.

$$\mathcal{D}^{+\triangledown}_{-\vartriangle}(t,r) \;=\; \frac{1}{4\pi r}\,\big(\;\delta(t-r) - \delta(t+r)\;\big) \qquad (1.60)$$

If we assume for now that the symmetric-in-time photon propagator corresponds to the symmetric-in-energy momentum space propagator. (A real symmetric function has a real symmetric Fourier transform).

$$\mathcal{F}\left\{ \frac{1}{2\pi}\ \delta(t^2 - r^2) \right\} \;=\; \frac{-2}{E^2 - p^2} \tag{1.61}$$

Which includes a factor two for the two light-cones. We can determine the Hilbert partner of this propagator via the Hilbert transform after expanding it.

$$\frac{-2}{E^2 - p^2} = \frac{1}{p}\left(\frac{1}{E+p} - \frac{1}{E-p}\right) \tag{1.62}$$

With the use of equation (1.57) we find for the Odd Hilbert partner:

$$\begin{aligned} \mathcal{D}^{+\nabla}_{-\Delta}(E,p) \;&=\; \mathcal{F}\left\{ \frac{1}{4\pi r}\ \Big(\ \delta(t-r) - \delta(t+r)\ \Big) \right\} \\ &= \mathcal{H}\left\{ \ +\frac{1}{p}\left(\frac{1}{E-p} - \frac{1}{E+p}\right) \right\} \\ &= -\frac{\pi}{ip}\Big(\delta(E-p) - \delta(E+p)\Big) \end{aligned} \tag{1.63}$$

We see that the Hilbert partner is its own Fourier transform (It is an eigen-function of the Fourier transform). It only propagates plane-waves which are "*on-shell*": Photons that have the right relation between energy and momentum. This means that it doesn't propagate "*virtual*" photons, which can have any relation. Such virtual photons occur if the source is a plane-wave of electric charge-current, for instance the interference pattern of an electron.

The Odd Hilbert partner and its Fourier transform differ only by a factor of $-4\pi^2 i$, which is $(\sqrt{2\pi})^4$ from the normalization factor $\sqrt{2\pi}$ of the physical Fourier transform applied 4 times for the 4d Fourier transform. The factor i is to be expected since the Fourier transform of an odd real function is an odd imaginary function.

Using $\frac{1}{2}(Even{+}Odd)$ = Forward propagator, we find for the causal forward in time photon propagator.

Forward in time Photon propagator

$$\mathcal{D}^{\triangledown}(E,p) \;=\; \frac{-1}{E^2-p^2} \;+\; \frac{\pi}{2ip}\Big(\delta(E+p)-\delta(E-p)\Big) \tag{1.64}$$

The latter part only propagates on-shell and doesn't alter the lower order approximations of Quantum Field Theory since real electrons can not emit real photons. For completeness: The backward in time propagator is found by using $\frac{1}{2}(Even - Odd)$. It flips the sign of the imaginary Dirac functions in the poles.

Backward in time Photon propagator

$$\mathcal{D}_{\vartriangle}(E,p) \;=\; \frac{-1}{E^2-p^2} \;-\; \frac{\pi}{2ip}\Big(\delta(E+p)-\delta(E-p)\Big) \tag{1.65}$$

We can reorganize the forward propagator on a pole-by-pole base to study the behavior at the poles:

$$\mathcal{D}^{\triangledown}(E,p) \;=\;$$
$$-\frac{1}{2p}\left(\frac{1}{E-p}-i\pi\delta(E-p)\right) \;+\; \frac{1}{2p}\left(\frac{1}{E+p}-i\pi\delta(E+p)\right) \tag{1.66}$$

Going through the pole we see that it first becomes $+\infty$ as a result of the reciprocal $1/(E-p)$ term, next it becomes $i\infty$ as a result of the delta term, and finally it becomes $-\infty$ because of the reciprocal term. The magnitude of the imaginary $i\infty$ term from the delta pulse is however infinitely larger as the peaks from the reciprocal term. To see this we can use Rayleigh's 'energy' theorem.

$$\mathbb{E}(f) \;=\; \int\limits_{-\infty}^{\infty} dt\ |f(t)|^2 \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\ |f(E)|^2 \tag{1.67}$$

Where $f(E)$ is the Fourier transform of $f(t)$ and 'energy' just means the sum of the squares of the absolute values. The *sum-of-squares* of a function and the sum-of-squares of its Fourier transform are the same up to a normalization constant. It follows that the 'energy' of the delta function is equal to the total sum-of-squares of the reciprocal term.

$$\frac{\pi^2}{4}\int\limits_{-\infty}^{\infty} dE\ \delta^2(E-p) \ = \ \int_{-\infty}^{\infty} dE\ \left|\frac{1}{E-p}\right|^2 \tag{1.68}$$

Since, in general, the sum-of-squares of a function is equal to the sum-of-squares of its Hilbert transform. The both are related to each other by a multiplication with the sign function in the Fourier domain.

Another compelling argument can be made that the addition of the delta functions is a physical requirement. Any real life photon has a finite lifetime and is therefor not exactly on-shell. Its frequency spectrum extends to both sides of the pole which propagate with opposite sign, and therefor, would destructively interfere.

The destructive interference would be 100% in the exactly symmetric case. Minimal changes in frequency would move the spectrum to either side of the pole and the destructive interference would disappear. The on-shell propagation would be ill-defined.

Only the addition of the delta functions at the poles makes the on-shell propagation well-behaved since the magnitude of the deltas is much higher as that of the reciprocal functions.

1.15 Propagators in the epsilon prescription form

The expression for the forward photon propagator separated into its two poles was derived in the previous section, equation (1.66)

$$\mathcal{D}^{\triangledown}(E,p) \ =$$
$$-\frac{1}{2p}\left(\frac{1}{E-p} - i\pi\delta(E-p)\right) \quad + \quad \frac{1}{2p}\left(\frac{1}{E+p} - i\pi\delta(E+p)\right) \tag{1.69}$$

This can alternatively be written using the socalled ϵ prescription as:

$$\mathcal{D}^{\triangledown}(E,p) \ = \ \lim_{\epsilon\to 0} \quad -\frac{1}{2p}\left(\frac{1}{E+i\epsilon\ -p} \quad + \quad \frac{1}{E+i\epsilon\ +p}\right) \tag{1.70}$$

In fact, one will generally will see the propagator only in the form of this ($\epsilon \rightarrow 0$) limit expression rather than the more elaborate Hilbert pair form we are using here. Combining the two poles in a single expression gives the standard textbook expression for the causal photon propagator.

Forward in time Photon propagator with ϵ prescription

$$\mathcal{D}^{\nabla}(E,p) \;=\; \lim_{\epsilon\rightarrow 0} \quad \frac{-1}{(E+i\epsilon)^2-p^2} \tag{1.71}$$

The backward in time photon propagator is obtained by changing the sign of the epsilon term.

Backward in time Photon propagator with ϵ prescription

$$\mathcal{D}_{\triangle}(E,p) \;=\; \lim_{\epsilon\rightarrow 0} \quad \frac{-1}{(E-i\epsilon)^2-p^2} \tag{1.72}$$

These propagators are real everywhere except in the poles $E^2 = p^2$ where the propagators become imaginary and infinite when ($\epsilon \rightarrow 0$)

The Sokhatsky Weierstrass theorem

The epsilon method is equivalent but not exactly the same. What can be said is that they lead to the same propagators in the position domain *after* the Fourier transform. The Fourier transform is in this case a special form of the so called Sokhatsky Weierstrass theorem:

If $f(x)$ is a complex-valued function which is defined and continuous on the real line, and the limits of the integral a and b are real constants with $a < 0 < b$, then

$$\lim_{\varepsilon\rightarrow 0^+}\int_a^b \frac{1}{x\pm i\varepsilon}\,f(x)\,dx \;=\; \int_a^b \left\{\,\frac{1}{x}\;\mp\; i\pi\delta(x)\,\right\} f(x)\,dx \tag{1.73}$$

So the equivalence is only valid in combination with the integration. We can see the difference if we use the fact that the two terms at the right hand side form a Hilbert pair and consequently should have the same "Energy" according to Rayleigh's theorem.

$$\int_{-\infty}^{+\infty} \left| \frac{1}{x} \right|^2 dx \quad = \quad \int_{-\infty}^{+\infty} \Big| \mp i\pi\delta(x) \Big|^2 dx \tag{1.74}$$

Furthermore both terms of the Hilbert pair should be equal or higher as the Rayleigh energy of the left hand side.

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \left| \frac{1}{x \pm i\varepsilon} \right|^2 dx \quad \leq \quad \int_{-\infty}^{+\infty} \left| \frac{1}{x} \right|^2 dx \tag{1.75}$$

Which tells us that the both sides are not exactly the same without the integration.

1.16 The Residue theory and Jordan's Lemma

The pole prescriptions find their origin in the theory of complex residues and contour integration, combined with Jordan's lemma which allows the calculation of certain Fourier transforms. We will briefly review the tools involved.

The Residue theorem

The residue theorem is a result of Cauchy's integral theorem. It states that the counter clockwise integral around a pole is:

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz \quad = \quad f(a) \tag{1.76}$$

Where $f(z)$ is any function which can be expanded into a Taylor series and the value $f(a)$ is called the residue. For higher order poles we get an expression which is sometimes called "*Cauchy's differentiation formula*", which is for an n'th order pole:

$$\frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz \quad = \quad f^{(n)}(a) \tag{1.77}$$

Here $f^{(n)}(z)$ is the n-th derivative of $f(z)$. If the integral is taken the other way around then the sign of the result is reversed. This because the values of dz along the contour are reversed.

Jordan's lemma

Jordan's lemma allows us to use the residue theorem to solve Fourier integrals. The Fourier transform is an integration from $-\infty$ to $+\infty$. We can turn this to a contour integral by closing it over a half circle with infinite radius. Jordan's lemma tells us that either the upper or the lower half circle integral is zero.

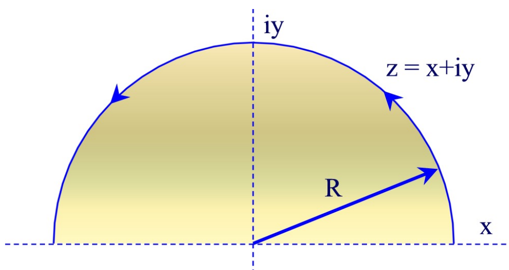

Figure 1.12: Jordan's lemma

Jordan's lemma tells us that either the upper or the lower half circle integral is zero in case of the Fourier transform.

$$\begin{aligned}\lim_{R\to\infty}\left|\oint_R e^{itz} f(z)\ dz\right| &= \lim_{R\to\infty}\left|\oint_R e^{iRt\ \mathrm{Re}(z)}\ e^{-Rt\ \mathrm{Im}(z)} f(z)\ dz\right| \\ &\leq \lim_{R\to\infty}\left|\oint_R e^{-Rt\ \mathrm{Im}(z)}\ |f(z)|\ dz\right| \end{aligned} \tag{1.78}$$

If the value z in $e^{i\omega z}$ contains a positive imaginary component then the infinite value R will suppress all contributions of $f(z)$ and the upper half circle becomes zero. Reversing the sign of t results in a zero lower half circle.

Fourier transform of the forward/backward photon propagators

We illustrate the application of the residue theorem by doing the first the *energy→time* Fourier transform of the forward photon propagator.

$$\mathcal{D}^{\triangledown}(t,p) \;=\; \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} dE\;\frac{-e^{-iEt}}{(E+i\epsilon)^2-p^2} \tag{1.79}$$

The $+i\epsilon$ prescription shifts the pole to just below the real axis. For positive t we have to close over the negative half circle in which case both poles are enclosed. For negative t we must close the contour over the positive half circle. Both poles are excluded which means that the contributions are zero for $t<0$, characteristic for a forward propagator.

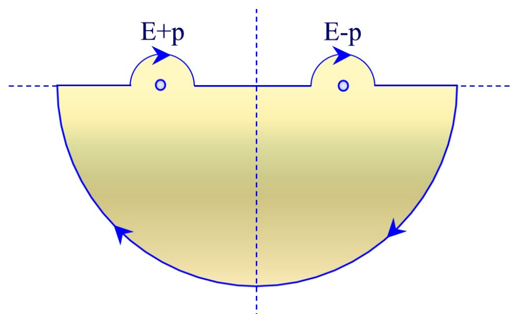

Figure 1.13: Forward propagator via the residue method

For the forward photon propagator we see from equation (1.70) for the separated poles, that the result, the two residues multiplied by $-1/(2p)$, becomes.

$$\mathcal{D}^{\triangledown}(t,p) \;=\; \left(+\frac{e^{ipt}}{i2p}\;-\;\frac{e^{-ipt}}{i2p}\right)\theta(t) \;=\; +\;\frac{\sin(pt)}{p}\;\theta(+t) \tag{1.80}$$

For the backward in time photon propagator we have to add $-i\epsilon$ instead. The result is now non-zero only for negative t for which we have to close the contour over the positive half circle. The sign is reversed because we now circle the poles in the opposite direction.

$$\mathcal{D}_{\triangle}(t,p) \;=\; \left(-\frac{e^{ipt}}{i2p}\;+\;\frac{e^{-ipt}}{i2p}\right)\theta(t) \;=\; -\;\frac{\sin(pt)}{p}\;\theta(-t) \tag{1.81}$$

1.17 Transform from position to momentum space

It is instructive to derive the causal photon propagator the other way around, via the Fourier transform from the position domain to the momentum domain. Starting with the derivation of the result for $\mathcal{D}^{\triangledown}(t,p)$ found above. We will use the spherical symmetry of the propagator to replace the 3d spatial Fourier transform.

$$\mathcal{F}^3(t,\vec{p}) \;=\; \iiint_{-\infty}^{+\infty} f(t,\vec{x})\; e^{i\vec{p}\cdot\vec{x}}\; d^3x \tag{1.82}$$

With the radial Fourier transform for 3d:

$$\mathcal{F}^3(t,p) \;=\; \frac{4\pi}{p}\int_0^{\infty} r\; f(t,r)\;\sin(pr)\; dr \tag{1.83}$$

The symmetric in energy propagator corresponds with the symmetric in time propagator.

$$\frac{-2}{E^2-p^2} \quad\Leftrightarrow\quad \frac{1}{2\pi}\;\delta(t^2-r^2) \tag{1.84}$$

Where the factor 2 in the nominator is the result of having both back and forward light-cones The symmetric in time propagator can be split in a forward propagator and a backward propagator.

$$\mathcal{D}^{+\triangledown}_{+\vartriangle}(t,r) \;=\; \frac{1}{4\pi r}\;\delta(t-r) \;+\; \frac{1}{4\pi r}\;\delta(t+r) \tag{1.85}$$

The radial Fourier transform gives us the propagators expressed in t and p. Due to the delta-function we simply have.

$$\mathcal{F}^3(t,p) \;=\; \frac{1}{p}\int_0^{\infty}\delta(t-r)\sin(pr)\; dr \;=\; \frac{\sin(pt)}{p} \tag{1.86}$$

Which gives us for the symmetric in time photon propagator.

$$\mathcal{D}^{+\triangledown}_{+\vartriangle}(t,p) \;=\; \frac{\sin(p|t|)}{p} \tag{1.87}$$

and for the anti-symmetric in time photon propagator.

$$\mathcal{D}^{+\nabla}_{-\Delta}(t,p) \;=\; \frac{\sin(pt)}{p} \tag{1.88}$$

The last step is to Fourier transform the time coordinate into the energy coordinate. For the symmetric in time/energy propagator we already know what the result is. The Fourier transform of the anti-symmetric propagator is trivial.

$$\int_{-\infty}^{+\infty} \frac{\sin(pt)}{p}\; e^{+iEt}\; dt \;=\; \frac{\pi}{ip}\Big(\delta(E+p) - \delta(E-p)\Big) \tag{1.89}$$

Which leads us to the pole modifying delta functions.

1.18 EM propagation and Huygens principle

In 1690, Christiaan Huygens proposed his famous Huygens principle for the propagation of light. It states that each point of an advancing wave front is in fact the center of a new disturbance and the source of a new front of waves, consequently the advancing wave as a whole may be regarded as the sum of all the secondary waves arising from all points in the medium where the wave front already passed. Huygens principle enabled physicist to understand most of the properties of wave propagation like diffraction and refraction.

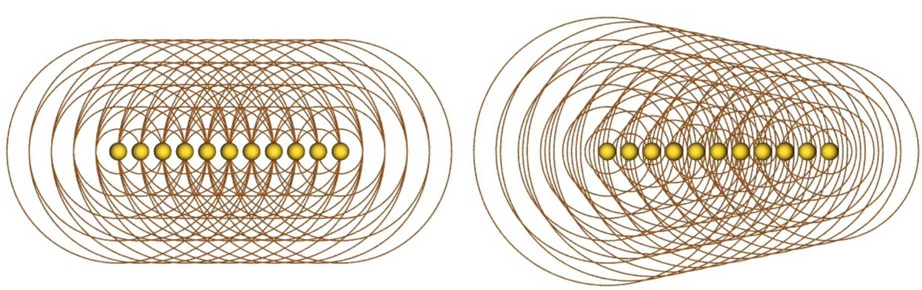

Figure 1.14: Huygens principle for charge/current sources

When it comes to the mathematical description of Huygens principle then it's quite common to discuss the Wave equation which we have studied extensively in this chapter. However, what Huygens principle describes is the propagation from *the field itself*, while the Wave equation we have discussed sofar describes the propagation from the *charge/current density* as a source of the electromagnetic field.

$$\Box\Phi \;=\; \frac{\rho}{\epsilon_o}, \qquad \Box A^i \;=\; \mu_o j^i \tag{1.90}$$

Reversing the d'Alembertian operator gives us.

$$\Phi \;=\; \Box^{-1}\left(\;\frac{\rho}{\epsilon_o}\;\right), \qquad A^i \;=\; \Box^{-1}\left(\;\mu_o j^i\;\right) \tag{1.91}$$

Where the reversed d'Alembertian is understood to be the (forward) propagator which spreads the disturbance spherically with the speed of c. The wave equation therefor can be used to analyze the resultant fields from oscillating electric sources as visualized figure (1.14) in a way which is very similar to Huygens principle.

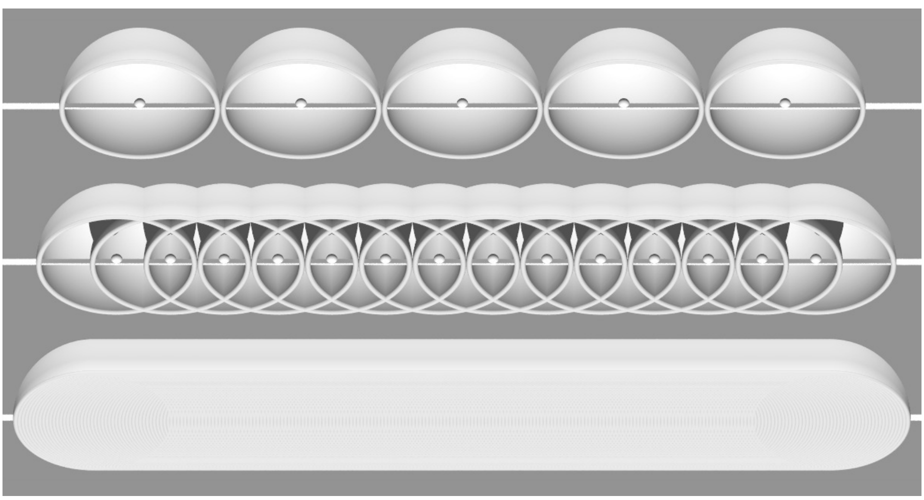

Figure 1.15: EM propagator for charge/current sources

This method is widely used to construct phase array radars and sonars. The direction of the wave front can be electronically controlled by introducing a phase shift over the array of sources.

The Wave equation as defined here can not be used to determine the correct self propagation of the electromagnetic field. From figure 1.14 we can also see the first problem. The wave front is symmetrical under horizontal reflection. This means that there would also be a wavefront propagating backward counter to the planewave.

Another even more fundamental problem becomes clear if we look at figure 1.15. The source in this case is a line which is non-zero only once at $t = 0$. Each point of the line is the origin of an outwards spreading sphere. The upper line of figure 1.15 shows how the delta function disturbance at a limited number of points resulting in hollow spheres propagating outwards from those points at the speed of light.

If we increase the number of spheres until they cover the entire line then we see that, instead of a "hollow" wave front propagating away at $v{=}c$, we actually get a "solid response" The field is non-zero everywhere within the outer shell. (This corresponds with the solid 1d (plane-wave) propagator as shown in figure 1.8 as we could expect).

1.19 The Huygens (self) propagator of the EM field

Going back to the math, from the d'Alembertian being zero in the vacuum one could draw the conclusion that the EM potentials are not their own source, with the rather compelling argument that the propagation is on the light cone which would be unlikely if light would be re-transmitted by the vacuum.

However, we nevertheless *can* construct a propagator, a Greens function, which does exactly what the Huygens principle intends to do. The simplest way to obtain this propagator is to start with the 1d version (the plane-wave propagator). If we have a plane wave at t given by $A(t)$ then the self-propagator has to shift the plane wave over a distance of $c\,dt$ after a time dt. So the Green's function for the one dimensional case is given by.

$$\mathcal{D}_1 \quad = \quad \delta(ct - r) \tag{1.92}$$

We can obtain the three dimensional case by applying the inter-dimensional operator to go from the 1d to the 3d radial propagator.

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{\pi}\,\frac{\partial}{\partial(s^2)}\,\mathcal{D}_1 \;=\; \frac{1}{\pi}\,\frac{\partial}{\partial(s^2)}\,\delta(ct-r) \tag{1.93}$$

The shifting delta function in 1d becomes the surface of an expanding sphere in the 3d case. The surface is differentiated along the normal and the time. Since $s^2 = c^2t^2 - r^2$ which we can split in derivatives in t and r.

$$\frac{\partial\psi}{\partial s^2} \;=\; \left(\frac{\partial\psi}{\partial t}\right)_r \frac{\partial t}{\partial s^2} \;+\; \left(\frac{\partial\psi}{\partial r}\right)_t \frac{\partial r}{\partial s^2} \tag{1.94}$$

$$= \left(\frac{\partial\psi}{\partial t}\right)_r \frac{\partial}{\partial s^2}\left\{\,\tfrac{1}{c}\sqrt{s^2+r^2}\,\right\} \;+\; \left(\frac{\partial\psi}{\partial r}\right)_t \frac{\partial}{\partial s^2}\left\{\,\sqrt{c^2t^2-s^2}\,\right\}$$

The contributions of these two terms are orthogonal to each other. The left one represents the contribution from varying t^2 and the other one the contribution from varying r^2. The propagator is symmetric in r and ct.

We can simplify it either to the t or r component because the propagator is non-zero only on the light-cone at r=ct, and we can relate the derivatives as $\partial_r = -\partial_{ct}$ which is true if they operate on any arbitrary function $f(ct-r)$.

The self-propagator of a mass less particle

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{2\pi ct}\,\frac{\partial}{\partial ct}\left\{\,\delta(ct-r)\,\right\} \;=\; -\,\frac{1}{2\pi r}\,\frac{\partial}{\partial r}\left\{\,\delta(ct-r)\,\right\} \tag{1.95}$$

The $-\frac{\partial}{\partial r}$ derivative operates on the expanding spherical surface in the radial direction. This amounts to a *positive* spherical surface at the outside and a *negative* spherical surface on the inside. The $\frac{\partial}{\partial t}$ derivative leads to the same propagator.

The propagator is the response on the 4d Dirac function $\delta(x^\mu)$. The positive outer spherical surface can be seen as a response of the Dirac function going "up". The negative inner spherical surface is then the result of the Dirac function going back down to zero.

As illustrated in figure 1.16, with white and black distinguishing between the signs. If we increase the number of sphere pairs to cover the entire plane, we see that the two types cancel everywhere within the time-like interval except at the edge, where the propagations is at c and where the negative sphere, which is slightly behind, can not cancel the positive sphere.

So, we see that the disturbance only propagates with c from a plane wave point of view. Figure 1.16 integrates over a 1d line. We extend the picture to 4d and define the disturbance as a plane-like delta function which propagates with time into the direction of the "wave-front".

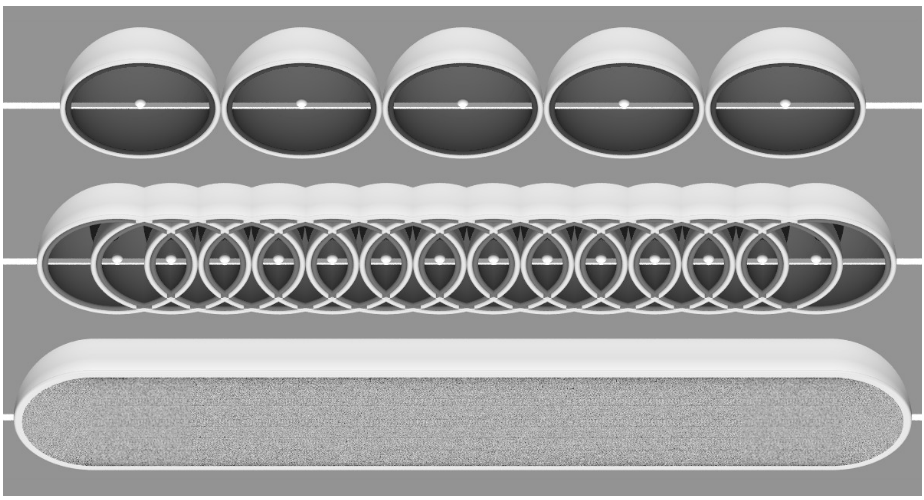

Figure 1.16: Huygens (self) propagator of the EM field

It turns out that the propagator also eliminates the backward moving wave-front. This is a result of the derivative in the in the propagator. The contribution coming from previous position of the "delta-plane" cancels the contribution from the current position.

If the light-front reaches a hole in the screen then the field in the hole will present a point source as in Huygens principle and the resulting field will again be spherical. The field will exclusively propagate on the light cone with all other slower paths canceled.

This cancelation ("interference") is an important property of the self-propagator and works for any kind of waveform, unrelated to any particular wave length.

The self propagator via the Fourier transform

Another way to determine the self-propagator is to use the fact that the spatial Fourier spectrum of a point contains all frequencies in equal amount. The propagation of the field in time is simply given by the sum of the propagation of all the frequencies as eigenfunctions of the field.

$$\mathcal{D}_3(t,r) \quad = \quad \int \frac{dp^3}{(2\pi)^3}\; \theta(t) \exp\Big(-\, iEt \;+\; ip^i x^i \Big) \tag{1.96}$$

The propagator above simplifies in the case of a the zero mass electromagnetic field where $E = c|p|$ to.

$$\mathcal{D}_3(t,r) \quad = \quad \int_{-\infty}^{\infty} \frac{dp^3}{(2\pi)^3}\; \theta(t) \exp\left(-ic|p|t\right)\, \exp\left(ip^i x^i \right) \tag{1.97}$$

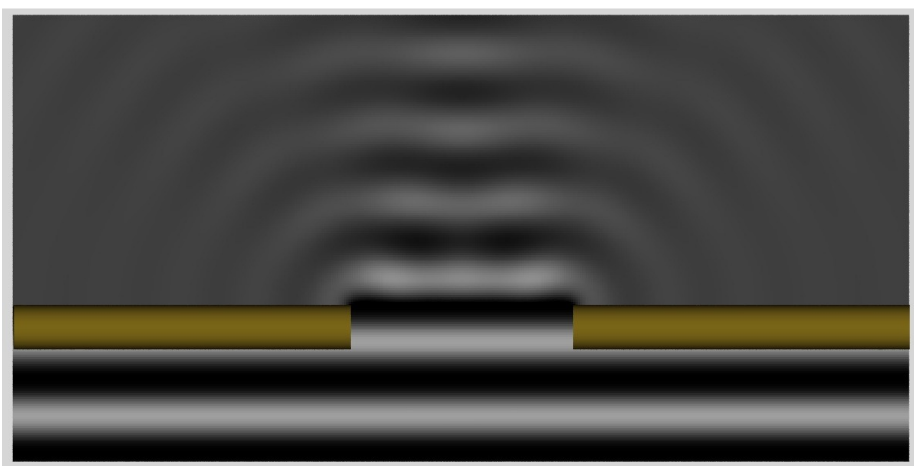

Figure 1.17: Diffraction of light behind a pinhole

We will show that we can derive the self propagator for a mass less particle from this expression. The expression represents a 3d Fourier transform. Since the propagator is radially symmetric we can simplify this to a 3d radial Fourier transform.

$$\mathcal{D}_3(t,r) \quad = \quad \frac{1}{\pi r}\int_0^{\infty} \frac{dp}{2\pi}\; \theta(t)\, p\, \exp\left(-icpt\right)\, \sin\left(pr \right) \tag{1.98}$$

We can solve this mathematically as a 1d Fourier transform like this.

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{\pi r}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\Big[\,\theta(t)\,p\,\sin\left(pr\right)\Big]\,e^{-i(ct)p} \tag{1.99}$$

Note that the this expression is symmetrically in p so we could expand the boundaries to $+\infty, -\infty$. This step doubles the result at one hand but it also halves the result by mapping equally to the $+t$ and $-t$ domains. Splitting the sine in exponentials gives us.

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{i2\pi r}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\Big[\,\theta(t)\;p\left(e^{+irp}-e^{-irp}\right)\Big]\;e^{-i(ct)p} \tag{1.100}$$

The factor p in the function between square brackets is equivalent to the derivative in ct in the Fourier transformed domain.

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{2\pi r}\,\frac{\partial}{\partial ct}\left\{\int_{-\infty}^{\infty}\frac{dp}{2\pi}\;\left[\,\theta(t)\;\left(e^{+irp}-e^{-irp}\right)\right]\;e^{-i(ct)p}\right\} \tag{1.101}$$

The remaining Fourier transform is elementary.

$$\mathcal{D}_3(t,r) \;=\; \frac{\theta(t)}{2\pi r}\,\frac{\partial}{\partial ct}\Big\{\,\delta(ct-r)\;-\;\delta(ct+r)\,\Big\} \tag{1.102}$$

The second term propagates backward in time and is discarded due to the Heaviside step function $\theta(t)$. This gives us the following 3d radial propagator (r>0)

$$\mathcal{D}_3(t,r) \;=\; \frac{1}{2\pi r}\,\frac{\partial}{\partial ct}\Big\{\,\delta(ct-r)\,\Big\} \tag{1.103}$$

Which is equivalent to the expressions in (1.95) since on the light cone we have ct=r and in general ∂_r=$-\partial_{ct}$ is true for any arbitrary function of the form $f(ct-r)$.