Chapter 9
Relativistic matter waves from Klein Gordon's equation
–
from my book:
Understanding Relativistic Quantum Field Theory

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Contents

Chapter 9

Relativistic matter waves from Klein Gordon's equation

9.1 The Klein Gordon Equation

We started in this book with the classical wave equation which was visualized in figure ??. the classical wave equation governs the propagation of massless fields at the speed of light. The addition of a mass term will now lead us to discuss the Klein Gordon equation for fields which have mass and which can propagate with any speed between plus c and minus c.

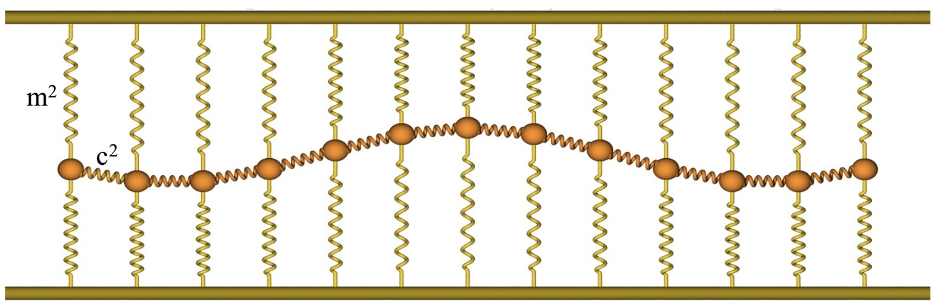

Figure 9.1: Mechanical equivalent of the (real) klein-Gordon equation

The mass term can be interpreted in the mechanical equivalent by springs which oppose the perturbation ψ of the weights in the vertical direction.

$$\text{Klein Gordon equation:} \qquad \frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = -\frac{m^2c^4}{\hbar^2}\,\psi \tag{9.1}$$

We first consider the case in which there is only a single set of m^2 springs at the end of the horizontal chain. Waves propagating at c towards the end will be reflected back negatively in the opposite direction. This becomes, with a continuous distribution of m^2 springs, a recursive process of reflections and reflections of reflections.

From equation (9.1) we can, in comparison with the d'Alembertian of the electromagnetic potentials, interpret the field as it's own (negative) source expressing the opposition of the vertical springs against the perturbation. To find an expression for the propagation we need to reverse (9.1) which tracks back to the source to obtain the Green's function which expresses the propagation resulting from a delta pulse like perturbation of the field. The equation for Green's function $D(t, r)$ is:

$$(\Box + m^2)\ D(t,r) \ = \ \delta(t,r) \tag{9.2}$$

Where we have used $c = 1$ and $\hbar = 1$ for simplicity and the d'Alembertian in 4d space-time is defined by.

$$\Box \ = \ \frac{\partial^2\psi}{\partial t^2} \ - \ c^2\frac{\partial^2\psi}{\partial x^2} \ - \ c^2\frac{\partial^2\psi}{\partial y^2} \ - \ c^2\frac{\partial^2\psi}{\partial z^2} \tag{9.3}$$

Expressing this in momentum space instead of position space gives us via the Fourier transform:

$$(-p^2 + m^2)\ D(E,p) \ = \ 1 \qquad \Rightarrow \qquad D(E,p) \ = \ \frac{-1}{p^2 - m^2} \tag{9.4}$$

Where p^2 is a shorthand notation of $(E^2 - p_x^2 - p_y^2 - p_z^2)$. This expression has two poles at $p^2 - m^2$ which is a result of the infinite plane wave character of the momentum space representation. The infinity at the pole only occurs after infinite propagation time when all contributions from infinitely far away have arrived. We can extract the recursive behavior of the propagator by expressing it as a geometric series.

$$\frac{1}{p^2 - m^2} \ = \ \frac{1}{p^2} + \frac{m^2}{p^4} + \frac{m^4}{p^6} + \frac{m^6}{p^8} + ... \tag{9.5}$$

Where the terms at the right hand side do not contain the pole anymore. We see that the first term is just the massless propagator. The second term can be interpreted as the first reflection/re-emission which is proportional to m^2, the third term represents the second reflection et-ceterea. When we Fourier transform the re-emission series back to positions space we get Green's function as a series in the d'Alembertian:

$$D(t,r) \ = \ \Box^{-1} - m^2\ \Box^{-2} + m^4\ \Box^{-3} - m^6\ \Box^{-4} + \ .... \tag{9.6}$$

All components of this series propagate on the lightcone. However, we now do have propagation *inside* the lightcone also due to the reflections in contrast with the massless propagator. This means that Klein Gordon particles with mass can move at any speed between plus c and minus c.

We can see from (9.6) that there is no propagation outside the light cone, preserving special relativity and causality. We have circumvented the unphysical infinity at the $p^2 = m^2$ poles with the use of the reflection series.

9.2 Relativistic de Broglie matter waves

The Klein Gordon equation leads us to the relativistic matter waves first proposed by Louis de Broglie which are fundamental in the sense that they govern the basic behavior of all types of matter observed sofar. Of elementary importance are the relations which link energy and momentum with the frequencies of the matter waves in time (9.7) and space (9.8)

These relations are at the origin not only of quantum mechanics, but also of special relativity. They, like the classical wave equation, lead to the rules of special relativity. The Klein Gordon equation and its descendants could be said to impose special relativity on nature, rather than merely being Lorentz invariant.

$$E = hf \tag{9.7}$$

The Energy is found to be proportional to the phase change rate in time given by the frequency f and Planck's constant h. This expression also implies the momentum / wavelength relation (9.8) if we assume special relativity. The momentum p is found to dependent on the phase change rate over space resulting in the deBrogle matter wavelength λ:

$$p = h/\lambda \tag{9.8}$$

The deBroglie wave-length is related one-to-one to the relativistic effect of non - simultaneity: The particle at rest has the same phase throughout its wave function. In regards with figure(9.6) this corresponds to a horizontal line of weights moving up and down.

Non-simultaneity as observed from other reference frames results in a different phase at different locations. The resulting phase shift over space becomes the de Broglie wave length. The deBroglie wave-length is a relativistic effect even though it does occurs at speeds of centimeters or less per second. $p = h/\lambda$ is not a separate law but is already implied by $E = hf$.

On the other hand, if we want the Klein Gordon equation to impose the laws of special relativity then it should naturally produce wave packages moving with the speed corresponding to the de Broglie wave length. Furthermore, it should produce the well known phase speed of $c^2/v > c$. And indeed, it does so, and it does so as well for our mechanical spring/mass model.

9.3 The wave packet at rest and moving

The wave function of a particle in its rest frame is represented by (9.9), where Q_x is a localized, real valued, Quantum wave packet. E_0 is the energy belonging to its rest mass m_0. The particle viewed from its rest-frame has an equal (complex) phase over all of space: This means that a particle at rest has a deBroglie wavelength λ of ∞.

$$\text{Particle at rest:} \qquad \Psi \;=\; Q_x\, e^{-i2\pi f\mathbf{t}} \;=\; Q_x\, e^{-iE_0\mathbf{t}/\hbar} \tag{9.9}$$

The Function of the wave-packet Q_x is to localize the particle. For the next few sections we will work with pure plane waves. We will include Q again at the section which discusses the group speed of the deBroglie wave.

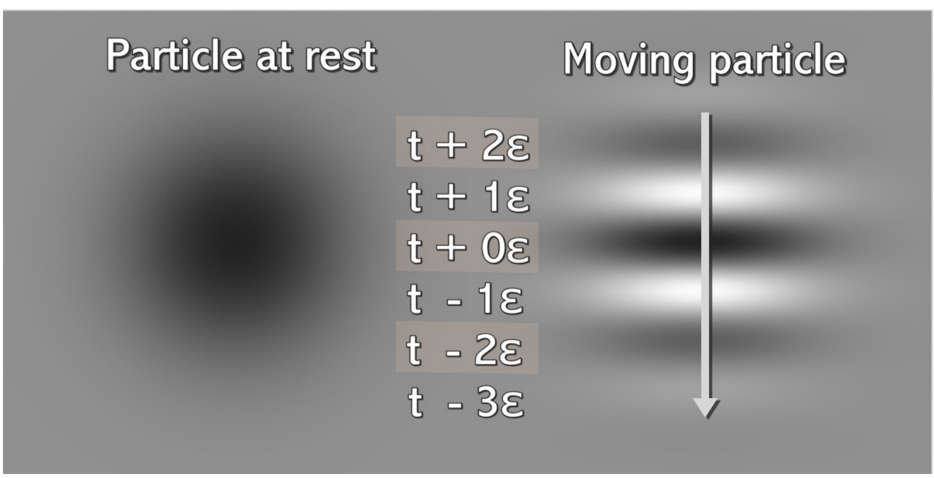

Figure 9.2: The de Broglie wave as a result of non-simultaneity

The relativistic time shift seen from a reference frame other than the rest frame produces different phase shifts in $e^{iE0t/\hbar}$ at different x locations which then manifest them self as the deBroglie wave length, a complex phase changing over space with a wave length λ.

$$\text{Moving:}\quad \Psi \;=\; e^{-i2\pi f\mathbf{t} \;+\; i2\pi\mathbf{x}/\lambda} \;=\; e^{\;-iE\mathbf{t}/\hbar \;+\; ip\mathbf{x}/\hbar} \tag{9.10}$$

We can simply derive the formula above from (9.9) if we substitute t with t' from the Lorentz transformation:

$$t' = \gamma\left(\, t - \frac{vx}{c^2} \,\right) \tag{9.11}$$

$$e^{-E_0\mathbf{t}/\hbar} \;=\; e^{-im_0c^2\mathbf{t}/\hbar} \Rightarrow e^{-im_0c^2\,\gamma\,(\,t - vx/c^2\,)\,/\hbar} \;=$$
$$=\; e^{-i\gamma m_0c^2t/\hbar + i\gamma m_0vx/\hbar} = e^{\;-iEt/\hbar \;+\; ipx/\hbar} \tag{9.12}$$

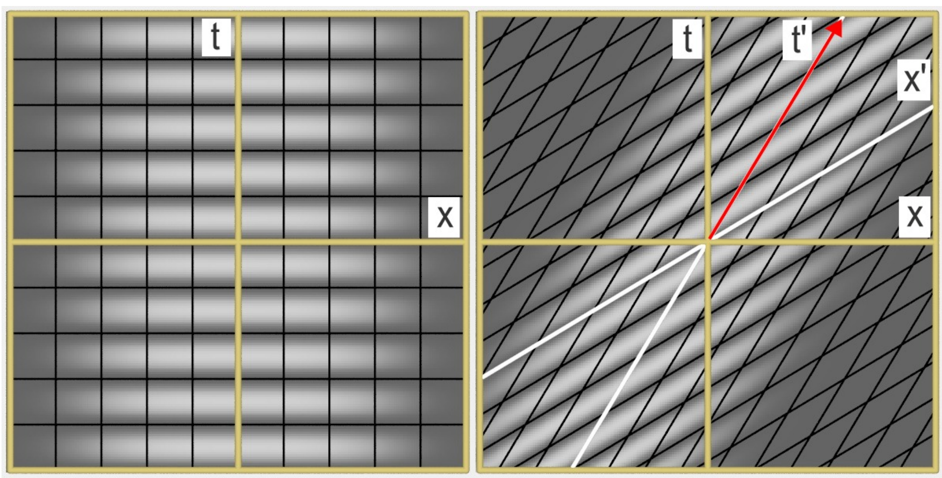

Figure 9.3: de Broglie waves of a particle at rest (left) and moving (right)

With the relativistic momentum $p = \gamma m_0 v$ and the relativistic energy $E = \gamma m_0 c^2$ we get our expression (9.10) for the moving particle. We have derived the wave behavior of momentum from the wave behavior of energy. Figure (9.2) shows a particle at rest with $\lambda = \infty$ (localized by the function Q) and a particle moving downwards with an indication of the time bands in the rest frame of the particle

9.4 de Broglie waves as particles and antiparticles

The relativistic de Broglie plane waves are the eigen functions of the Klein Gordon equation which is easy to show by inserting the solutions in the Klein Gordon equation. We obtain the classical energy momentum relation.

$$E^2 - c^2p^2 \;=\; m^2c^4 \tag{9.13}$$

There are however two different solutions which we'll associate with particles and antiparticles.

$$\text{particles} \;:\; e^{\;-iEt/\hbar \;+\; ipx/\hbar} \tag{9.14}$$

$$\text{antiparticles} \;:\; e^{\;+iEt/\hbar \;+\; ipx/\hbar} \tag{9.15}$$

Note that our solutions are complex and that we want to give a physical meaning to the complex pair. In general this kind of relationship is interpreted as a position/momentum relationship whereby the position trails the momentum by 90 degrees. In the anti-particle case this relation is reversed. Alternatively, one can interpret the complex pair as the two coordinates of the plane of rotation whereby the rotation can be both left and right handed.

In any case, one does somehow expect a direction in space to be associated with the oscillating variation. This however isn't the case with the scalar Klein Gordon particles. Furthermore, only a single (phase) parameter is not enough to specify a direction.

Existing scalar Klein Gordon particles are all composite particles where the spins are opposite and cancel. Thus, the spins define a direction in space for the individual particles but not do so anymore for the composite spin zero Klein Gordon particles.

9.5 The relativistic rotation of the wave front

A remarkable amount of physics can be extracted from the simple rule that the wave front is always at right angles with the physical velocity, regardless of the reference frame. This gives us another means of determining the material speed.

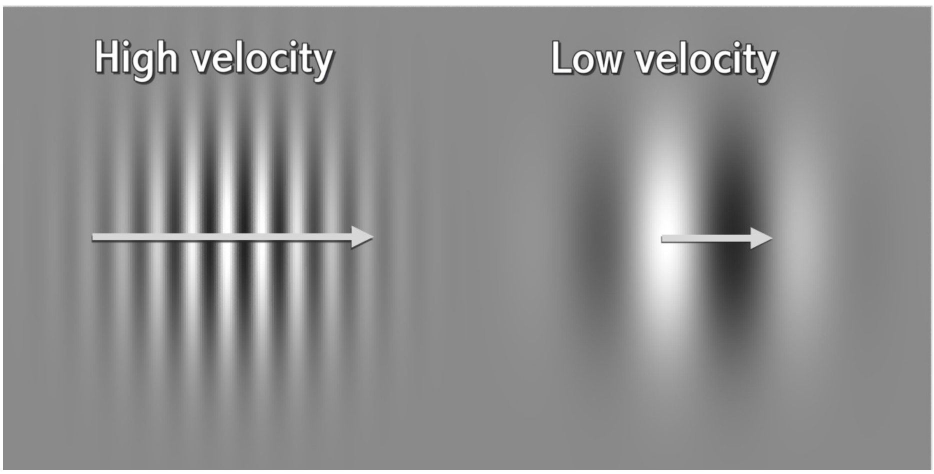

Figure 9.4: A faster particle chasing a slower one

The left half of image (9.4) shows a fast particle chasing a slower particle at the right with equal mass. The fast particle has a shorter de Broglie wavelength. The phase speed of the faster particle is slower (as given by c^2/v) compared to the slower particle.

In image (9.5) we see the same scene but now from a reference frame moving upwards. The extra motion has a larger influence on the slower moving particle. Its relative motion changes downwards more than the faster particle. As one can see, the combination of Special Relativity and Quantum Mechanics makes sure that the wavefronts are exactly at right angles with the physical speed, exactly as one would intuitively expect.

It is only Special Relativity which can rotate wavefronts, and it does so for both light and matter waves. A Galilean transformation keeps the wavefronts always directed in the same direction! The mechanism through which Special Relativity manages this is again via the non-simultaneity of time.

The time in the moving frame has progressed further in the upper time bands and less in the lower. Horizontally the phase has shifted further in the upper and less in the lower bands. The result is that the wavefront becomes skewed. The wavefront of the slower particle which has a higher phase speed (c^2/v) becomes more skewed and rotates further. Just as it should be to keep the wavefront at right angles with physical speed.

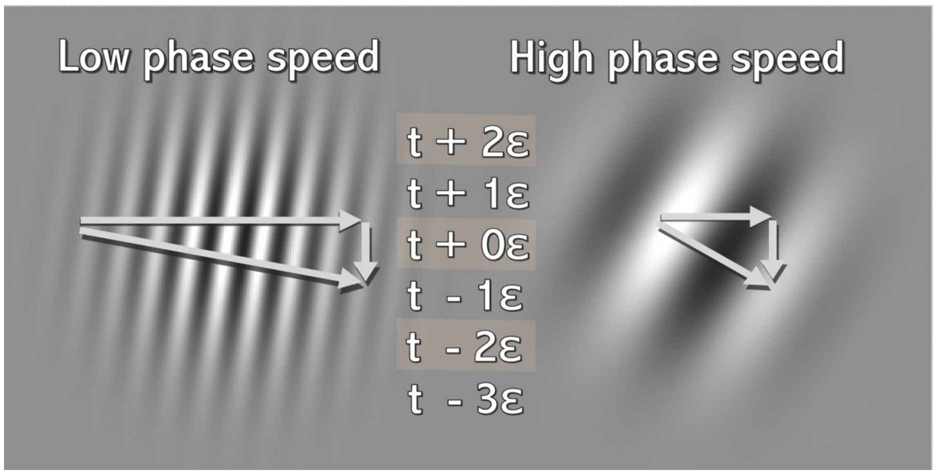

Figure 9.5: The scene in a different reference frame with a vertical speed

So it's Special Relativity which rotates the wavefront while it is the Quantum Mechanical de Broglie wave with its phase speed of c^2/v which rotates the wavefront of a slow particle more than that of a faster one. This mechanism works equally well for light waves which represent the limit where: group-speed = phase-velocity = c.

9.6 Lorentz contraction and Time dilation

The de Broglie wave length is inversely proportional to the speed and becomes infinite in the rest frame. Infinite simply means that the phase is equal everywhere in the rest frame. A small change in phase could be interpreted as a shift of the sinusoidal wave function over a distance which is infinite in the limit. The "speed" with which the phase shifts in the rest frame ($= f\lambda$) would thus be "infinite". The phase speed is the inverse of the material speed v. The phase speed only equals the material speed in the limit of c:

$$\text{de Broglie phase speed:} \quad v_\psi \;=\; f\lambda \;=\; \frac{E}{p} \;=\; \frac{c^2}{v} \tag{9.16}$$

This result, logical after we have seen the derivation of $p = h/\lambda$ from $E = hf$ does of course not contradict special relativity. We want to see what happens at the material speed v and start with:

$$\text{Moving wave packet:} \quad Q_{(x,t)} \;\; e^{+\, ipx/\hbar} \;\; e^{-iEt/\hbar} \tag{9.17}$$

Which we have split into three parts. First we want to express the localized packet Q more explicitly as a something which moves with a speed v and hence is Lorentz contracted by a corresponding gamma. We do so by defining:

$$Q_{(x,t)} \;\;=\;\; Q(\gamma(x - vt)) \tag{9.18}$$

Now we want to do the same for the second part of equation (9.17) which describes the phase change over space. We want to make it move with a physical speed v so we can view Q and the second term as a single combination which moves along with speed v. We already have the gamma factor included since:

$$e^{ipx/\hbar} \;\;=\;\; e^{i\gamma m_0 vx/\hbar} \tag{9.19}$$

(See equation (9.12), To make it physically moving at speed v we need to lend some from the third term to obtain the $-vt$ part of the factor $(x-vt)$. To do so we split the exponent of the third term as follows:

$$-iE\;t/\hbar \;=\; -\;iE\;t/\hbar\left(\frac{v^2}{c^2}\right) \;-\; iE\;t/\hbar\left(1-\frac{v^2}{c^2}\right) \tag{9.20}$$

The first half we move to the space phase so we get:

$$e^{ipx/\hbar}\;\; e^{-iEt/\hbar} \;\;\Rightarrow\;\; e^{im_0 v\gamma(x-vt)/\hbar}\;\; e^{-\;im_0c^2t/(\hbar\gamma)} \tag{9.21}$$

With this we can write the re-arranged expression (9.17) for the moving wave packet:

$$\begin{aligned} Q(\gamma(x-vt)) \quad e^{ip_0\gamma(x-vt)/\hbar} \quad e^{-\,iE_0t/(\hbar\gamma)} \quad &= \\ = \quad W(\gamma(x-vt)) \quad e^{-\,iE_0t/(\hbar\gamma)} & \qquad (9.22) \end{aligned}$$

Where W is the combined Lorentz contracted function moving with speed v. The phase variation with time represented by the last factor is now to be understood as taken over the actual trajectory of the wave packet. It correctly corresponds with the time dilation, which is predicted by special relativity to be a factor γ slower as for the particle at rest.

Both the Lorentz contraction (with factor gamma) and the phase variation with x are the result of the non-simultaneity of Special Relativity. To see this we can imagine that we instantaneously "freeze" a bypassing traveler. Walking around him we can see him "hanging in the air", indeed being contracted in the direction in he was moving.

The traveler however will complain that his front was stopped first, before his back was frozen, and argues that this is the reason of his compressed state. The same is true for the phase. The phase of the traveler does not vary with x in his rest frame. However since (as seen from his rest frame) we froze his front first and his back later. We end up with the phase variation over x given by the second part of equation (9.22)

9.7 Lorentz contraction from the Klein Gordon eq.

We want to find a localized wave packet Q for a free particle which is stable in time, that is, doesn't spread (disperses) or otherwise changes in time. We'll solve this first for the simplest relativistic equation, the Klein-Gordon equation. Furthes on we will extend this to the Dirac and Proca equations.

$$E^2 \;-\; p^2c^2 \;-\; (m_0c^2)^2 \;=\; 0 \qquad (9.23)$$

The classical relation for the relativistic particle above is the base of the Klein Gordon equation which yields the Schrödinger equation in the non-relativistic limit.

$$\text{Klein Gordon:} \qquad \hbar^2\frac{\partial^2\psi}{\partial t^2} \;-\; c^2\hbar^2\frac{\partial^2\Psi}{\partial x^2} \;+\; (m_0c^2)^2\psi \;=\; 0 \qquad (9.24)$$

We separate Ψ as below where Q is a localized Quantum wave packet which has a constant energy E and moves with a fixed momentum p. The E and p values here correspond to the classical center values.

$$\psi \quad \equiv \quad Q_{xt}\ (e^{i2\pi\mathbf{x}/\lambda})\ (e^{-i2\pi f\mathbf{t}}) \quad \equiv \quad Q_{xt}\ e^{ip\mathbf{x}/\hbar - iE\mathbf{t}/\hbar} \tag{9.25}$$

Note that we have assumed that the spread in p can be neglected. The spread in p becomes higher if the wave-packet becomes more localized. The second order derivative in time becomes written out:

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{\hbar^2}\left\{ E^2 + \frac{2i\hbar E}{Q}\frac{\partial Q}{\partial t} - \frac{\hbar^2}{Q}\frac{\partial^2 Q}{\partial t^2}\right\}\psi \tag{9.26}$$

Since we want Q to be a constant localized function which shifts along with physical speed v we can express the derivatives in time as derivatives in space:

$$Q(\gamma(x-vt)) \quad \text{gives:} \quad \frac{\partial Q}{\partial t} = -v\frac{\partial Q}{\partial x}, \qquad \frac{\partial^2 Q}{\partial t^2} = v^2\frac{\partial^2 Q}{\partial x^2} \tag{9.27}$$

Which is valid for any non-changing wave packet Q moving at a constant velocity v. We will use these identities to make our equation time independent and write for the partial derivatives:

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{1}{\hbar^2}\left\{ E^2 - \frac{2i\hbar Ev}{Q}\frac{\partial Q}{\partial x} - \frac{\hbar^2 v^2}{Q}\frac{\partial^2 Q}{\partial x^2}\right\}\psi \tag{9.28}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{1}{\hbar^2}\left\{ p_x^2c^2 - \frac{2i\hbar p_x c^2}{Q}\frac{\partial Q}{\partial x} - \frac{\hbar^2 c^2}{Q}\frac{\partial^2 Q}{\partial x^2}\right\}\psi \tag{9.29}$$

$$\frac{\partial^2}{\partial y^2}\psi = -\frac{1}{\hbar^2}\left\{ -\frac{\hbar^2 c^2}{Q}\frac{\partial^2 Q}{\partial y^2}\right\}\psi \tag{9.30}$$

$$\frac{\partial^2}{\partial z^2}\psi = -\frac{1}{\hbar^2}\left\{ -\frac{\hbar^2 c^2}{Q}\frac{\partial^2 Q}{\partial z^2}\right\}\psi \tag{9.31}$$

We then insert these terms in the Klein Gordon equation. The first order derivative terms cancel each other since $Ev = pc^2 = mc^2v$. The remaining terms become:

$$E^2 - p_x^2c^2 - p_y^2c^2 - p_z^2c^2 = m^2c^4 - \frac{\hbar^2c^2}{Q}\left[\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}\right] \tag{9.32}$$

We then insert these terms in the Klein Gordon equation. The first order derivative terms cancel each other since $Ev = pc^2 = mc^2v$. The remaining terms become:

$$E^2 - c^2p_x^2 \;=\; m^2c^4 \;-\; \frac{\hbar^2c^2}{Q}\left[\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 Q}{\partial z^2}\right] \tag{9.33}$$

That is, the moving packet Q is compressed by a factor γ in the x direction. The second order derivatives are higher by a factor γ^2 which is canceled by the factor in final formula.

The crucial step is replacing the derivatives in time by the derivatives in space for a stable solution shifting along with speed v. This is in essence what causes Lorentz contraction. We see that the Lorentz contracted Laplacian has to be zero in order to retrieve the classical relativistic particle equation.

9.8 Sub-luminal propagation of the field

The most far reaching consequence of the introduction of the mass term in the Klein Gordon equation is that it allows fields to propagate at other speeds as the speed of light. All speeds are possible within a range of.

$$-c \;<\; v \;<\; +c \tag{9.34}$$

Excluding c itself. We will study the wave-based mechanism of motion with the help of the momentum domain, see figure 9.6. It shows a momentum with a certain spread Δp around a central momentum.

Once we know the momentum distribution at a certain time, then we can determine the progression in time because the frequency component of a certain momentum has to be either $\pm\,\omega_p$, where $\omega_p = \sqrt{(pc)^2 + (mc^2)^2}/\hbar$

The (free field) solutions for particles have $-\omega_p$ for particles and $+\omega_p$ for anti-particles. In figure 9.6 we can follow the progression in time of the momentum distribution. The Δp of the momentum distribution stays the same but the phase changes according to the ω_p of the particular momentum.

The result is that a phase change rate builds up along the momentum-axis.

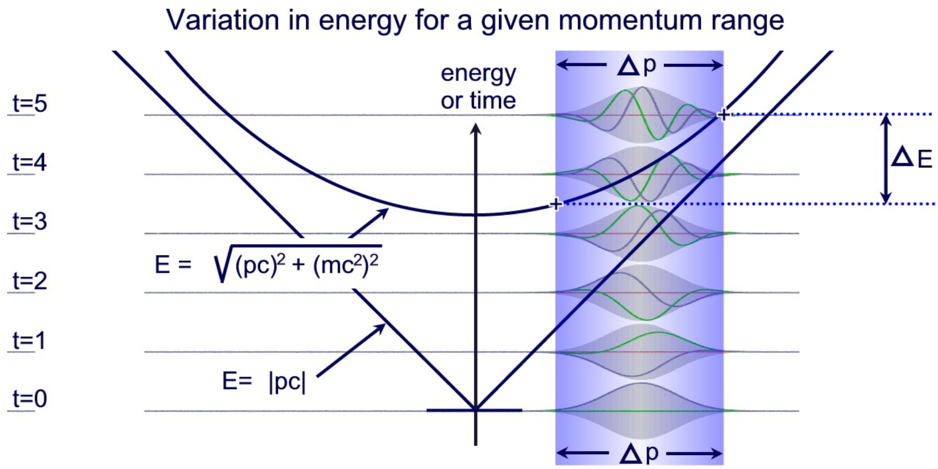

Figure 9.6: Variation in energy versus variation in momentum.

Such a phase change over the momentum-axis corresponds with a *translation* in the position domain, see figure 9.7 which shows both position and momentum domains.

Since the phase change rate increases linearly with time, we see a corresponding linear increase in the translation of the wave-packet: It has a constant velocity. Both figures show the real (blue) and imaginary (green) values of the field.

The velocity of the packet is proportional to the phase change rate which is in first order approximation given by the slope of the energy curve.

$$v \;=\; \frac{\partial E}{\partial p} \;=\; \frac{pc^2}{\sqrt{(pc)^2+(mc^2)^2}} \;=\; \frac{pc^2}{E} \tag{9.35}$$

Which we can check via relativistic classical mechanics.

$$\frac{pc^2}{E} \;=\; \frac{\gamma\, mv\, c^2}{\gamma\, mc^2} \;=\; v \tag{9.36}$$

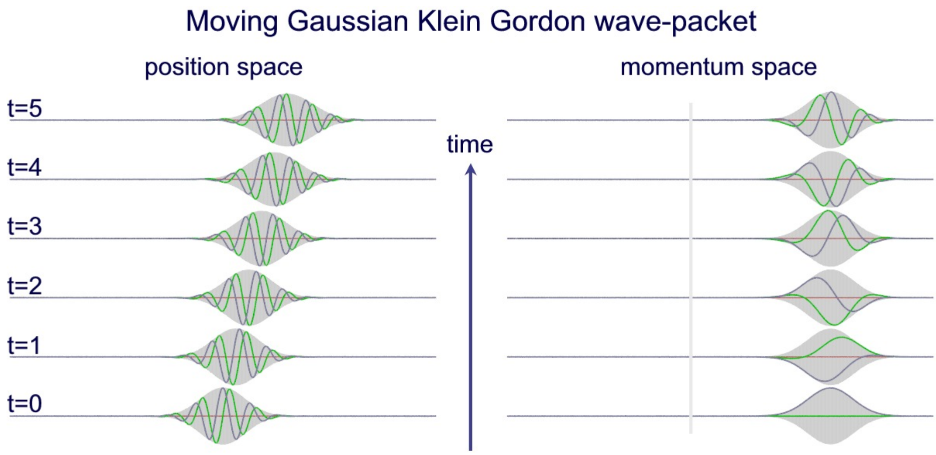

Figure 9.7: Moving Gaussian Klein Gordon wave-packet

Here we have retrieved the canonical equation, from classical mechanics, for deriving the velocity from the Hamiltonian, which was known long before relativistic quantum mechanics did explain its geometric origin.

$$v \quad = \quad \frac{\partial H}{\partial p} \tag{9.37}$$

The slope of the hyperbolic energy curve is not constant, except for the case where m=0. In the massless case there is only one speed possible.

$$v \quad = \quad \frac{\partial E}{\partial p} \quad = \quad \frac{\partial}{\partial p}\left\{\, pc \,\right\} \quad = \quad c \tag{9.38}$$

The phase change rate is uniform along the momentum-axis in this case, so the whole field in position space is translated with a uniform value and the form of the wave-packet does not change in time.

In the case of a particle with mass this is not the case. The result is that the shape of the field in position space changes in time.

9.9 Spreading of the free field wave packet

We discussed how the speed of the wave packet is given by the derivative of the Hamiltonian against the momentum.

$$v \;=\; \frac{\partial H}{\partial p} \;=\; \frac{\partial E}{\partial p} \;=\; \frac{pc^2}{\sqrt{(pc)^2+(mc^2)^2}} \;=\; \frac{pc^2}{E} \tag{9.39}$$

The wave-packet would not spread in the case of a constant v such as in the case of a massless particle which is represented with a wave-function which moves unchanged at the speed of light.

However, for a localized field p will vary and $E=\sqrt{(pc)^2+(mc^2)^2}$ means that there will be a range of speeds which means in general that a wave-packet will spread. The variation is approximately given by the first derivative.

$$\Delta v \;\approx\; \frac{\partial^2 E}{\partial\, p^2}\,\Delta p \tag{9.40}$$

Given that Heisenberg's uncertainty relation $\Delta x\Delta p \geq \hbar/2$ can be derived by Fourier analysis, which in the case of a Gaussian shaped wave-function becomes $\Delta x\Delta p = \hbar/2$, the minimum value, we can write.

$$\Delta v \;\approx\; \frac{\partial^2 E}{\partial\, p^2}\,\frac{\hbar}{2\Delta x} \tag{9.41}$$

Where Δx is the width. One can reason that the overall *shape* of a wave-function changes faster if Δx is smaller for a given speed-variation Δv. We can define a dimension-less quantity *shape*, which has derivative in time which gives us an approximation of the relative spreading of the wave-function in time.

$$\frac{\partial}{\partial t}\Big\{\text{shape}\Big\} \;\approx\; \frac{\Delta v}{\Delta x} \;\approx\; \frac{\partial^2 E}{\partial\, p^2}\,\frac{\hbar}{2(\Delta x)^2} \tag{9.42}$$

Working out the second order derivative gives us.

$$\frac{\partial^2 E}{\partial\, p^2} \;=\; \frac{(mc^2)^2\; c^2}{\big(\;(pc)^2+(mc^2)^2\;\big)^{3/2}} \;=\; \frac{E_o^2\, c^2}{E^3} \;=\; \frac{c^2}{E\gamma^2} \tag{9.43}$$

Which leads us to our final expression here.

$$\frac{\partial}{\partial t}\Big\{\text{shape}\Big\} \;\approx\; \frac{\hbar\, c^2}{2E(\gamma\Delta x)^2} \tag{9.44}$$

If we remove the gamma's then we get the expression for the rest frame.

$$\frac{\partial}{\partial t}\Big\{\text{shape}\Big\} \;\approx\; \frac{\hbar\, c^2}{2mc^2(\Delta x)^2} \tag{9.45}$$

We can summarize the results as:

- The spreading of the wave-function is inversely proportional to the frequency (the phase change rate in time) of the particle, Higher mass particles spread slower.
- The spreading of the wave-function is proportional to the square of the momentum spread. The smaller the initial volume in which the initial wave-function was contained the faster it spreads and keeps spreading.

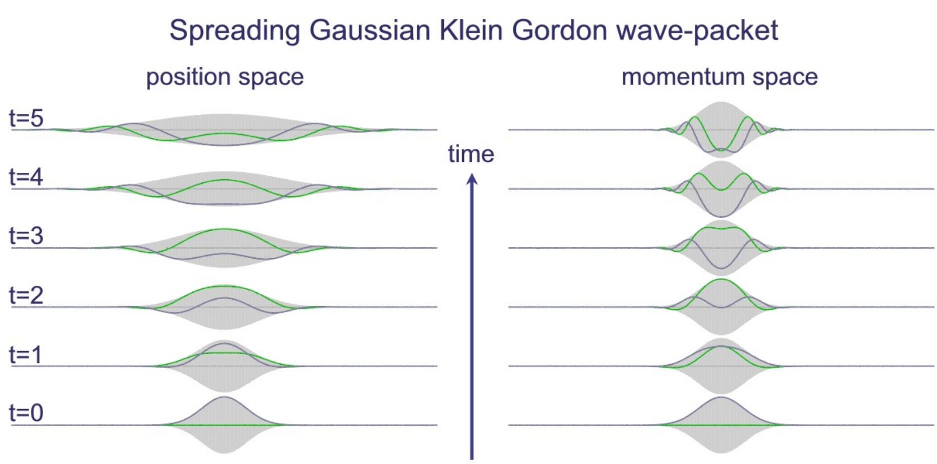

Figure 9.8: Spreading Gaussian Klein Gordon wave-packet

From figure. 9.8 we can read the mathematical mechanism which leads to spreading. The variation Δp of the momentum stays the same over time. It is the frequency dependency on the momentum $E = \sqrt{(pc)^2 + (mc^2)^2}$ which leads to a phase change over p.

The phase change is opposite at both sides of the center-momentum. These phase-changes lead to (opposite) translations of the wave-function in position space, this is the spreading. The value Δx in our expression stays constant because Δp stays constant, it is the initial Δx corresponding to the pure Gaussian at t=0.

Some actual spreading rate numbers

We can work out a few numerical example to get an idea of the spreading rates. From the wide range of wavelength sizes, we can classify the Compton radius as the low end, although there is in principle no real barrier to go to even smaller sized wave-packets.

If we replace Δx with twice the Compton radius $r_c = \hbar/mc$ then, assuming that our proximation is still reasonably valid in this range.

$$\frac{\partial}{\partial t}\Big\{\text{shape}\Big\} \quad \approx \quad \frac{c}{8\, r_c} \tag{9.46}$$

If we recall the rest-frequency of the particle: $f_o = c/(2\pi\, r_c)$ (in case of the electron f_o = 1.2355899729 10^{20} Hz), then we see that spreading speed approaches the speed of light in this range. The spread in momentum is so large that it includes velocities from close to $-c$ up to $+c$.

To confine an electron-field to a Compton radius-like volume one needs a positive charge of 137e, The inner-most electrons of heavy elements come close to being confined into such a small area. The Compton radius for electrons is 3.861592696 10^{-13} meter.

More commonly, electrons freed from a bound state, take off with a much larger radius, comparable to the Bohr radius. (5.291772131 10^{-11} meter) This means that the spreading speed is much lower, $v < 0.01c$, but still quite high.

The size of the wave-packet will grow fast. For instance, the famous single electron interference experiment of Akira Tomomura, which demonstrated the single-electron build-up of an interference pattern, shows that the electron fields in the experiment must at least be several micrometers wide. This is a factor 100,000 wider as in the confinement of the Bohr radius.

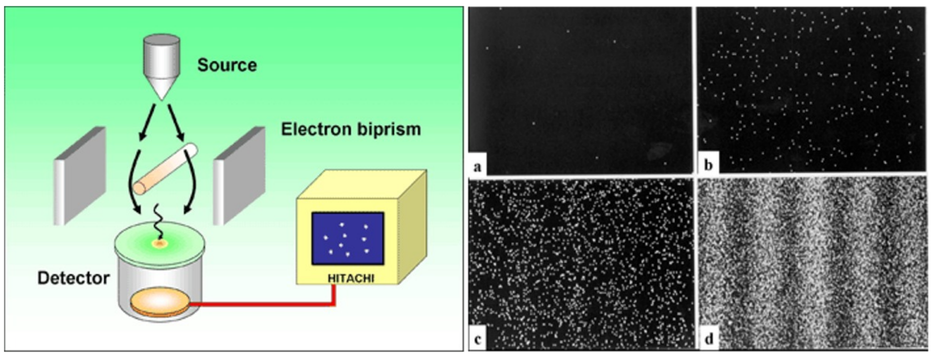

Figure 9.9: Akira Tomomura's experiment, www.hitachi.com