Chapter 11
EM Lorentz force derived from Klein Gordon's equation
—
from my book:
Understanding Relativistic Quantum Field Theory

Hans de Vries

February 2, 2010

Contents

Chapter 11

EM Lorentz force derived from Klein Gordon's equation

11.1 Klein Gordon equation with EM interaction

From classical electrodynamics we know that the total energy-momentum in an electromagnetic field A^μ is given by.

$$P^\mu \;=\; \left\{ \;\frac{1}{c}\,E,\;\vec{P}\;\right\} \;=\; \left\{\;\gamma mc+\frac{e}{c}\,\Phi,\quad \gamma m\vec{v}+e\vec{A}\;\right\} \tag{11.1}$$

This total four-momentum is the sum of the inertial four-momentum due to its mass *plus* the four-momentum due to the interaction with the electromagnetic 4-potential.

It is called the canonical momentum or the conjugate momentum because it is conjugate to the position in quantum mechanics. That is, it is this momentum which determines the phase change rates in space and time.

We did see that for the Klein Gordon field we can obtain the total, (local) conjugate momentum from the field ψ as follows.

$$\begin{aligned} p^o &= \quad \frac{i\hbar}{2}\left(\;\psi^*\frac{\partial\psi}{\partial x^o}-\frac{\partial\psi^*}{\partial x^o}\psi\;\right) \;=\; -\hbar\;\frac{\partial\phi}{\partial x^o}\;\psi^*\psi \\ p^i &= -\;\frac{i\hbar}{2}\left(\;\psi^*\frac{\partial\psi}{\partial x^i}-\frac{\partial\psi^*}{\partial x^i}\psi\;\right) \;=\; \hbar\;\frac{\partial\phi}{\partial x^i}\;\psi^*\psi \end{aligned} \tag{11.2}$$

The subtraction guarantees that only the phase change rates from $\phi(x^\mu)$ of $\psi = \exp(\,i\phi(x^\mu) + a(x^\mu)\,)$ give a contribution to the momentum and not the magnitude factor $a(x^\mu)$. The above equations are the result from symmetrically applying the derivative operator to both ψ and ψ^*.

To separate the electromagnetic interaction momentum from the total momentum we must replace the derivative operator with one which subtracts the additional phase change rates from electromagnetic interaction. This operator is given by the replacement.

$$\partial_\mu\phi \quad\Longrightarrow\quad (\;\partial_\mu - ieA_\mu\;)\,\phi \tag{11.3}$$

Applying this operator instead of the normal derivative in the form for the general operator,

$$\mathcal{O} \;=\; \psi^*\,\overleftrightarrow{\tilde{\mathcal{O}}}\;\psi \;=\; \Big(\;\psi^*\tilde{\mathcal{O}}\;\psi \;\;+\;\; \psi\;\tilde{\mathcal{O}}^*\psi^*\;\Big) \tag{11.4}$$

then gives us the expressions for the inertial 4-momentum, the momentum due to the mass only.

$$
\begin{aligned}
p^o &= \quad \frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial x^o} - \frac{\partial \psi^*}{\partial x^o}\psi \right) - \frac{e}{c}\,\Phi\ \psi^*\psi \\
p^i &= -\ \frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial x^i} - \frac{\partial \psi^*}{\partial x^i}\psi \right) - e\,A^i\ \psi^*\psi
\end{aligned}
\qquad (11.5)
$$

The inertial momentum p^μ_m transforms like a four vector while the 4-potential A^μ does so as well. They represent four degrees of freedom per point. On the other hand, The phase ϕ of the field ψ is a scalar, with only one degree of freedom per point. This means that the total canonical momentum is proportional to the gradient of the phase ϕ.

This restricts the degrees of freedom of the vector and thus put a restriction on the *sum* of the inertial momentum p^μ and the electromagnetic interaction momentum $p^\mu_e = eA^\mu\ \psi^*\psi$.

This means that if A^μ changes in a way that would make the total canonical momentum incompatible with the gradient from a scalar, then p^μ_m has to change as well to compensate for this. Thus a change in A^μ necessitates a change of the inertial momentum. This change of inertial momentum is of course the Lorentz force.

We can say that this restriction to a scalar phase imposes a $U(1)$ symmetry upon the Klein Gordon field, which is basically a field with an electric charge (but without a spin and thus without a magnetic moment). The group $U(1)$ contains all values given by $\exp(i\phi)$ where ϕ is real.

gauge covariant derivative

We can extend the free (interaction-less) Klein Gordon equation to the interaction Klein Gordon equation as follows,

$$
\partial_\mu \partial^\mu \psi + \left(\frac{m^2c^2}{\hbar^2}\right)\psi = 0 \quad \Longrightarrow \quad D_\mu D^\mu \psi + \left(\frac{m^2c^2}{\hbar^2}\right)\psi = 0 \qquad (11.6)
$$

Where D_μ is known as the *gauge covariant derivative*, and defined as

$$
D_\mu = \partial_\mu - i\left(\frac{e}{\hbar}\right)A_\mu, \qquad D^\mu = \partial^\mu - i\left(\frac{e}{\hbar}\right)A^\mu \qquad (11.7)
$$

11.2 Aharonov Bohm effect and experiments

The influence of the electromagnetic potential A^μ on the phase change rates of the wave-function was well known in the early days of quantum mechanics. The interacting Klein Gordon equation in the first section is in fact Dirac's starting point in his 1928 paper where he introduces his famous Dirac equation.

Nevertheless, somehow the physical importance of the potentials, rather then only the E and B fields, wasn't wider spread to the broader community. Illustrative is that two papers of Aharonov and Bohm in 1959 [**?**], and 1961 [**?**], written three decades later, led to the association of their names with the effect of A^μ on the phase change rates, which from then on was known as the Aharonov-Bohm effect.

The magnetic Aharonov Bohm effect

Most of the interference experiments specifically showed the influence of the magnetic vector potential on the phase change rates. Figure 11.1 shows how an electron wave function passes a solenoid at two sides through areas with opposing vector potentials. A change in the current through the solenoid leads to a change in the phase difference which then causes a change of the interference pattern on the detector plate.

Typically the diverging beams are bent back together simply with the help of two negatively electrically charged plates (an electron bi-prism).

The first experiment in 1960 by R.G. Chambers actually used a tiny magnetic iron whisker instead of a solenoid. A year later Möllenstedt and W.Bayh had developed a machine for the fabrication of very fine coils with diameters from $5\,\mu m$ to $20\,\mu m$ which they used for experiments confirming the effect.

In the context of the experiment it was important to show that the effect didn't follow from the magnetic field B but purely from the vector potential A^i. The effect should be there even if the magnetic field B is zero everywhere along the path of the wave-function. This is what leads us to the use of a solenoid because of it's $1/r$ like potential outside the solenoid:

$$\vec{A}_{ext} \;=\; \frac{1}{2r}\mu_o n R_a^2 I \vec{\phi} \;=\; \frac{1}{2}\mu_o n R_a^2 I \left(-\frac{y}{r^2}\,\vec{x} \;+\; \frac{x}{r^2}\,\vec{y}\right) \qquad (11.8)$$

The curl of this field yields zero, and thus $B = 0$ everywhere outside the solenoid. Any dependency on the current I through the solenoid thus demonstrates that A^i has a physical meaning in quantum mechanics. The solenoids radius is noted by R_a and n is the number of windings. The delta phase induced by the magnetic vector potential is given by.

$$\Delta\phi \;=\; \frac{e}{\hbar}\int_P \vec{A}\cdot d\vec{x} \tag{11.9}$$

The electron does not accelerate because $B = 0$. Thus, there is no change in the phase change rates due to the inertial momentum p^i_m. The measured $\Delta\phi$ is purely due to vector potential A^i.

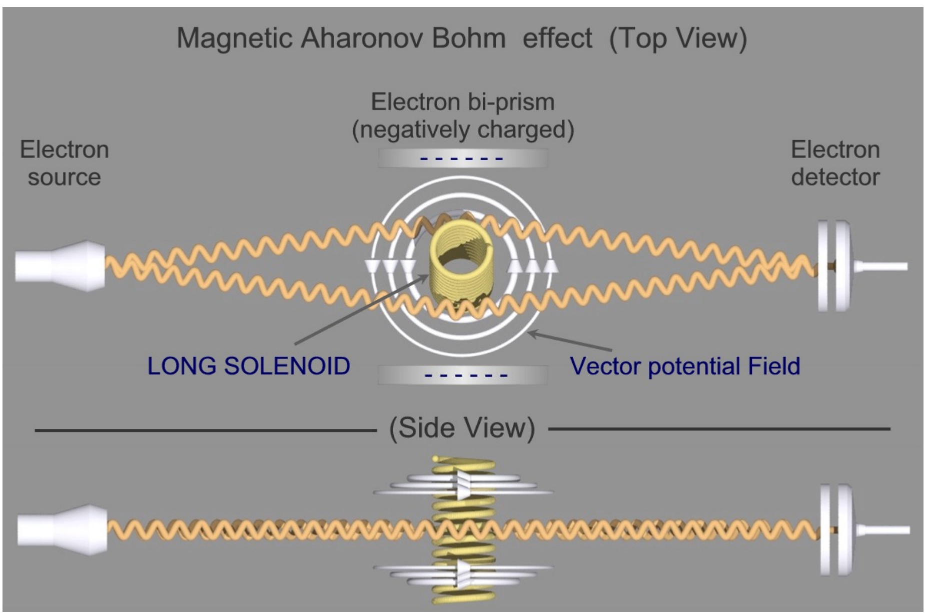

Figure 11.1: Top and side view of the experiment

Stokes law tells us that we can also express the $\Delta\phi$ as the integral over the enclosed surface of the component of B normal to the surface.

$$\Delta\phi \;=\; \frac{e}{\hbar}\iint_S \mathsf{B}\cdot d\mathbf{S} \tag{11.10}$$

This means that somewhere B must differ from zero. This is the case inside the solenoid. The magnetic field B inside the long solenoid and the total flux through the inside surface, the integral of B over the surface, is.

$$\mathsf{B} \;=\; \frac{1}{2}\mu_o nI \qquad \Longrightarrow \qquad \Phi_B \;=\; \pi r^2 \mathsf{B} \;=\; \pi r^2 nI \tag{11.11}$$

According to Stokes law, the line integral of a vector field along a closed curve is equal to the surface integral of its curl over the surface enclosed by the curve. Dividing the flux Φ_B by the length $2\pi r$ of the curve gives the vector potential A^i. Thus, at the inside ($r{<}R_a$) of the solenoid the vector potential is expressed by.

$$\vec{A}_{int} \;=\; \frac{1}{2}\mu_o n\; rI\vec{\phi} \;=\; \frac{1}{2}\mu_o nI\; \Big(-\, y\; \vec{x} \;+\; x\; \vec{y} \Big) \tag{11.12}$$

The enclosed flux Φ_B doesn't increase anymore outside the solenoid corresponding with expression (11.8)

The magnetic Aharonov-Bohm effect was further shown experimentally in 1986 by Tonomura et al.[**?**] in a beautiful experiment showing a quantized phase shift between paths inside and outside a super conducting toroidal ring. Webb et al.[**?**] in 1985 demonstrated Aharonov-Bohm oscillations in ordinary, non-superconducting metallic rings. Bachtold et al. detected the effect in 1999 in carbon nanotubes.

The electric Aharonov Bohm effect

In the electric Aharonov Bohm effect, it is the 0^{th} component of A^μ, the electric scalar potential which is responsible for the phase shift. The total phase shift is an integral over t during the period over which the charge stays in the potential field Φ.

$$\Delta\phi \;=\; -\frac{e}{\hbar}\int \Phi\; dt \tag{11.13}$$

An experiment by Oudenaarde et al. in 1998 [**?**], using a ring structure interrupted by tunnel barriers, with a bias voltage V between the two halves of the ring, demonstrated the electric Aharonov-Bohm phase shift.

11.3 The scalar phase and Wilson Loops

We did see that the total (canonical) momentum of an electro magnetically interacting Klein Gordon field is determined by the phase *geometry* of the field.

$$P^\mu \;=\; \frac{i\hbar c}{2m}\Big(\, \psi^*\partial^\mu\psi - \psi\partial^\mu\psi^* \,\Big) \tag{11.14}$$

The total phase change rate is the sum of those from the inertial, mass related, momentum p^μ, plus the phase change rate due to the electromagnetic interaction.

$$P^\mu \;=\; p^\mu \;+\; e\,A^\mu \tag{11.15}$$

A combination of multiple adjacent charges of equal sign will increase the effective mass as well as the effective momentum at a given velocity.

The phase change rates are derived from the scalar phase ϕ. In general a closed loop contour integral, tracking the total phase change over the contour, is a multiple of 2π.

$$\oint \mathrm{grad}\,\phi(\,\vec{r}\,)\cdot ds \;=\; 2\pi n \tag{11.16}$$

In the context of the discussion here we will generally refer to such a contour integral as a Wilson loop. In the infinitesimal limit of $r{\rightarrow}0$ we drop the $n{\neq}0$ cases around orbital angular momentum singularities and define.

$$\lim_{r\rightarrow 0}\;\; \frac{1}{2\pi r}\oint \mathrm{grad}\,\phi(\,\vec{r}\,)\cdot ds \;=\; 0 \tag{11.17}$$

We do so assuming that the wave function ψ itself is always zero at singular points, so a loop integral of ψ around the singularity will always yield 0.

Equation (11.17) demonstrates an inherent property of a scalar field. This in contrast to loop integrals of vector fields which are unrestricted.

$$\begin{aligned} \oint \vec{p}_m \cdot ds &= \text{unrestricted} \\ \oint e\vec{A} \cdot ds &= \text{unrestricted} \end{aligned} \tag{11.18}$$

This means that the identity $P^\mu = p^\mu + eA^\mu$ imposes severe restrictions on the values that the vector fields can take. Changes in A^μ must be compensated by changes in p^μ in a way which respects the scalar nature.

Effectively this means that changes in A^μ necessarily result in acceleration. This acceleration now corresponds to the Lorentz force:

$$\vec{F} \;=\; \frac{\partial p^i}{\partial t} \;=\; e\Big(\;\mathsf{E} + \vec{v} \times \mathsf{B}\;\Big) \tag{11.19}$$

The aim of this chapter is to show in considerable detail that the Lorentz force is the result of the interacting Klein Gordon equation which describes a particle without magnetic moment (spin).

In a four dimensional space-time we can distinguish between a total of six surfaces in which we can define elemetary Wilson loops. See figure 11.2.

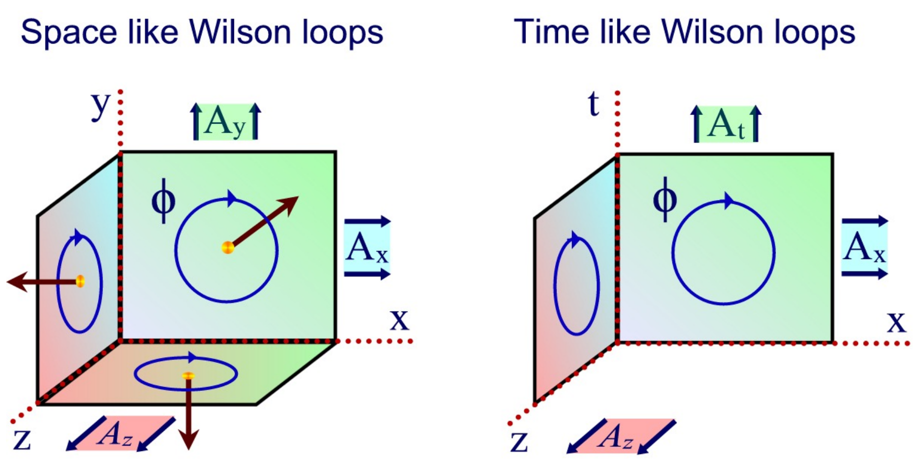

Figure 11.2: Space like and Time like Wilson loops

These six surfaces are simply the 6 permutations of the 4 dimensions. Note that we can define time-like loops as well which integrate over paths involving t. We can express infinitesimal loop integrals with the help of differential operators, for example.

$$\partial_y A^x - \partial_x A^y \quad = \quad \leftrightarrows \; + \; \downarrow\uparrow \quad = \quad \circlearrowleft \tag{11.20}$$

From $\psi = \exp(-iEt + ipx)$ we see that the phase shift rate in time is different in sign as the phase shift rates over space. With this in mind we can define all possible infinitesimal Wilson loops as.

$$F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu \tag{11.21}$$

This is the well known Faraday tensor of the electromagnetic field which we can write out explicitly as $F^{\mu\nu} =$

$$\begin{pmatrix} 0 & \partial_t A_x + \partial_x A_t & \partial_t A_y + \partial_y A_t & \partial_t A_z + \partial_z A_t \\ -\partial_x A_t - \partial_t A_x & 0 & \partial_x A_y - \partial_y A_x & \partial_x A_z - \partial_z A_x \\ -\partial_y A_t - \partial_t A_y & \partial_y A_x - \partial_x A_y & 0 & \partial_y A_z - \partial_z A_y \\ -\partial_z A_t - \partial_t A_z & \partial_z A_x - \partial_x A_z & \partial_z A_y - \partial_y A_z & 0 \end{pmatrix}$$

The Faraday tensor is anti-symmetrical with a zero diagonal, due to the subtraction in equation (11.21). The six independent infinitesimal Wilson loops determine the six components of the electromagnetic field.

$$F^{\mu\nu} \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.22}$$

(Where c is set to 1) This is the first step in deriving the Lorentz force from the interacting Klein Gordon equation. It identifies the quantities involved but it does not yet defines the Lorentz force in a unique way. The latter requires the combination of the special theory of relativity and the conservation of the 4-vector potential A^μ.

11.4 Lorentz force from the acceleration operator

We derived the Klein Gordon *acceleration operator* in the chapter on the "Operators of the scalar Klein Gordon field". We will apply this operator on the Klein Gordon field, which now will include the electromagnetic interaction terms, in order to derive the Lorentz force.

The acceleration operator was derived by applying the Hamiltonian $\tilde{H}$ twice on the position operator $\tilde{X}$. We will briefly recall the derivation here.

$$\tilde{\mathtt{A}}^i \;=\; -\frac{1}{\hbar^2}\Big[\,\Big[\,\tilde{\mathtt{X}}^i\,,\,\tilde{H}\,\Big]\,,\,\tilde{H}\,\Big] \;=\; -\frac{1}{\hbar^2}\Big[\,\tilde{\mathtt{X}}^i\,,\,\tilde{H}^2\,\Big] \qquad (11.23)$$

Two cross-terms did cancel in the leftmost expression. The squared Hamiltonian $\tilde{H}^2$ follows directly from the Klein Gordon equation itself.

$$\tilde{H}^2 \;=\; -\,\hbar^2\frac{\partial^2}{\partial t^2}\psi \;=\; \Big(\,-\,\hbar^2c^2\nabla^2\psi\,+\,m^2c^4\,\Big)\psi \qquad (11.24)$$

The terms of the Hamiltonian squared which do not commute with $\tilde{X}^i$, (and thus contribute to the acceleration operator), are the second order derivatives over the axis x^i corresponding with the i-th component of the position operator $\tilde{X}^i$.

$$\tilde{\mathtt{A}}^i \;=\; -\,\frac{1}{\hbar^2}\left[\,\frac{i\hbar}{2mc}\,x^i\,\frac{\partial}{\partial x^o},-\hbar^2c^2\left(\frac{\partial}{\partial x^i}\right)^2\,\right] \;=\; -\frac{i\hbar c}{m}\,\frac{\partial}{\partial x^o}\,\frac{\partial}{\partial x^i} \qquad (11.25)$$

This operator needs to be applied on the field $\psi = \exp(a + i\phi)$ in such a way that the derivatives do not pick up the real part a of the exponent. It is the phase ϕ which determines the momentum and its derivative in time.

For convenance we define the field ψ with phase only. The phase depends on the inertial momentum p^μ, due to the particle's mass, as well as the electromagnetic four-vector A^μ

$$\psi \;=\exp\left\{-\frac{i}{\hbar}\int\Big(\,p^o + eA^o\,\Big)dx^o \;+\; \frac{i}{\hbar}\sum_{i=1}^{3}\int\Big(\,p^i + eA^i\,\Big)dx^i\right\} \qquad (11.26)$$

This expression does contain integrals over space. However, Special Relativity does not allow us to do physical integrals over space: A change in A^i somewhere far away would change the phase instantaneously over the whole integrated area, violating the speed of light limitation.

This leads us to an essential rule: A change in A^i somewhere must be locally compensated by an equivalent change in p^i, so a change in A^i does not result to an immediate change in the phases. This happens only over time due to a change in p^o, the energy, which changes due to the change in the momentum p^i.

The phase ϕ in (11.26) corresponds to the total canonical momentum, while we are looking for $\partial p^i/\partial t$, the change of the inertial momentum.

The electric Lorentz force

We now turn back our attention to the acceleration operator (11.25). The order in which we apply the derivatives doesn't matter, so.

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^o}\psi \;=\; \frac{\partial}{\partial x^o}\frac{\partial}{\partial x^i}\psi \tag{11.27}$$

Applying the above on the field defined in (11.26) gives us the expression.

$$\frac{\partial}{\partial x^o}\Big(\; p^i + eA^i\;\Big)\psi \;=\; -\,\frac{\partial}{\partial x^i}\Big(\; p^o + eA^o\;\Big)\psi \tag{11.28}$$

These (three) expressions correspond with the three time-like Wilson loops.

We assume that the energy p^o, the phase change rate in time, is spatially constant, $\partial p^o/\partial x^i = 0$), and there is therefor no acceleration without the electromagnetic field. Using $x_o = ct$ and $\Phi = cA^o$ we can write.

$$\frac{\partial p^i}{\partial t} \;=\; -\,e\frac{\partial A^i}{\partial t} - e\frac{\partial \Phi}{\partial x^i} \tag{11.29}$$

We recognize the righthand side as the electric field.

$$\frac{\partial \vec{p}}{\partial t} \;=\; e\,\mathsf{E} \tag{11.30}$$

The magnetic Lorentz force

We have obtained the electric part of the Lorentz force. In order to obtain the magnetic part also we must assume that the field has a local effective velocity $\vec{v}$ which can be derived via the velocity operator.

For any arbitrary function f, over which the particle moves along with a velocity $\vec{v}$, we can write the derivative in time as.

$$\frac{df}{dt} \;=\; \frac{\partial f}{\partial t} \;+\; \frac{\partial f}{\partial x}v_x \;+\; \frac{\partial f}{\partial y}v_y \;+\; \frac{\partial f}{\partial z}v_z \tag{11.31}$$

This expresses that a moving particle will experience a spatial derivative as a temporal derivative. Replacing f with p^μ gives us

$$\frac{dp^\mu}{dt} \;=\; \frac{\partial p^\mu}{\partial t} \;+\; \frac{\partial p^\mu}{\partial x}v_x \;+\; \frac{\partial p^\mu}{\partial y}v_y \;+\; \frac{\partial p^\mu}{\partial z}v_z \tag{11.32}$$

The first term on the right is associated with the electric force and the latter three with the magnetic force. So for the full Lorentz force we have to consider the purely spatial variants of (11.27) also:

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\psi \;=\; \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\psi \qquad (i \neq j) \tag{11.33}$$

With ψ defined as in the integral expression (11.26) this gives.

$$\frac{\partial}{\partial x^i}\Big(\, p^j + eA^j \,\Big)\psi \;=\; \frac{\partial}{\partial x^j}\Big(\, p^i + eA^i \,\Big)\psi \tag{11.34}$$

If we single out one i,j-pair of the above expression, for instance x, y

$$\frac{\partial}{\partial x}\Big(\, p^y + eA^y \,\Big)\psi \;=\; \frac{\partial}{\partial y}\Big(\, p^x + eA^x \,\Big)\psi \tag{11.35}$$

and look at one component of the velocity, $v = (v_x, 0, 0)$, then this becomes.

$$\left(\, \frac{\partial p^y}{\partial t} \;+\; ev_x\frac{\partial A^y}{\partial x} \,\right)\psi \;=\; \left(\, 0 \;+\; ev_x\frac{\partial A^x}{\partial y} \,\right)\psi \tag{11.36}$$

This gives us a single component of the magnetic force in the y-direction.

$$\frac{\partial p^y}{\partial t} \;=\; ev_x\left(\frac{\partial A^x}{\partial y}-\frac{\partial A^y}{\partial x}\right) \;=\; -\,ev_x\,\mathsf{B}^z \tag{11.37}$$

Replacing x with z gives us the other component in the y-direction.

$$\frac{\partial p^y}{\partial t} \;=\; ev_z\left(\frac{\partial A^z}{\partial y}-\frac{\partial A^y}{\partial z}\right) \;=\; +\,ev_z\,\mathsf{B}^x \tag{11.38}$$

Further repeating this for the other combinations gives us the full magnetic Lorentz force $\vec{F} = e\vec{v}\times\vec{B}$

$$\begin{aligned} F^x &= \frac{\partial p^x}{\partial t} = ev_y\,\mathsf{B}^z - ev_z\,\mathsf{B}^y \\ F^y &= \frac{\partial p^y}{\partial t} = ev_z\,\mathsf{B}^x - ev_x\,\mathsf{B}^z \\ F^z &= \frac{\partial p^z}{\partial t} = ev_x\,\mathsf{B}^y - ev_y\,\mathsf{B}^x \end{aligned} \tag{11.39}$$

Parallel and orthogonal Lorentz force components

Instead of separating the Lorentz force into an electric and magnetic part we can also treat it as a combination of parallel and orthogonal components. The parallel components of F^x are those containing A^x.

Parallel Lorentz force terms

$$\Big(F^x\Big)_{\parallel} \;=\; -\,e\,\frac{\partial A^x}{\partial t} \;-\; ev_x\,\frac{\partial A^x}{\partial x} \;-\; ev_y\,\frac{\partial A^x}{\partial y} \;-\; ev_z\,\frac{\partial A^x}{\partial z} \tag{11.40}$$

Special relativity requires that the phase ϕ induced in the x-direction by the parallel components is directly compensated by a changing inertial momentum in the x-direction.

Orthogonal Lorentz force terms

$$\Big(F^x\Big)_{\perp} \;=\; -\,e\,\frac{\partial \Phi}{\partial x} \;+\; ev_x\,\frac{\partial A^x}{\partial x} \;+\; ev_y\,\frac{\partial A^y}{\partial x} \;+\; ev_z\,\frac{\partial A^z}{\partial x} \tag{11.41}$$

All terms are all derivatives in x. Notice how the term $\partial A^x/\partial x$ in both expression cancels for the total force F^x in the x-direction.

11.5 Parallel electric Lorentz force term

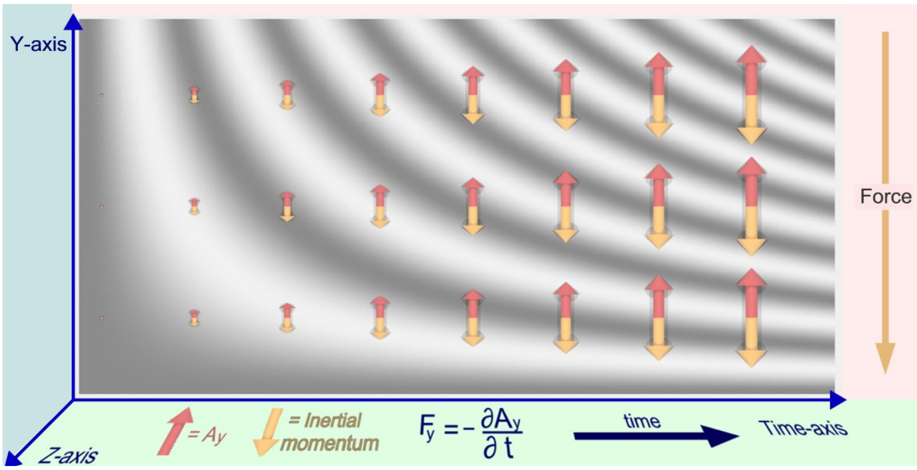

Figure 11.3: Electric force from a change of A_y in t

We can distinguish between four different types of contributions to the Lorentz force, these are handled in this section plus the next three ones. The phase pattern induced by the different ∂A^μ terms is the same in all four cases and shown as a gray background pattern.

Figure (11.3) shows the parallel electric term $-\partial_t A^y$. The time evolution is shown along the horizontal axis. An increasing A^y induces an increasing spatial phase change rate in the vertical direction.

Special relativity requires that the field ψ cancels this phase change rate by accelerating in the opposite direction because of locality. It has to be canceled because otherwise the local phase would depend on arbitrary far away values of $eA^y\psi$.

This component is the reason that an accelerating charge exerts a force on nearby charges in the opposite direction, thereby making it harder to accelerate a bunch of charges as a whole. The result is that a bunch of charges moving at a certain speed has a higher momentum than the inertial momentum based on the sum of the masses. It has an extra electromagnetic momentum.

11.6 Orthogonal electric Lorentz force term

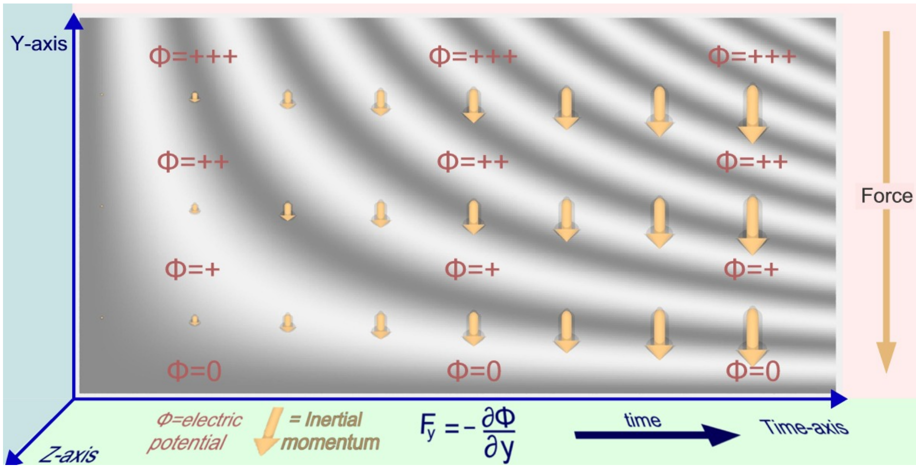

Figure 11.4: Electric force from a gradient of Φ in y

Figure (11.4) shows the orthogonal electric term $-\partial_y A^o$. There is a higher potential field ($\Phi = +++$) at the upper side of the image as compared to the bottom where ($\Phi = 0$). The time evolution is shown on the horizontal axis going to the right.

The differences of the phase change rates in time, at the top and at the bottom, build up an increasing spatial phase change rate along the y-axis. The field ψ has to cancel this spatial phase change rate with an opposing spatial phase change rate by accelerating along the y-axis.

The cancelation is required by locality and special relativity. It guarantees that arbitrary far away values of $e\Phi\psi$ do not have an instantaneous influence on the value of the local phase ϕ of the field, since the phase ϕ is co-determined by the integral $\int e\Phi\psi\, dy$.

This component is behind the observed fact that bound states have a certain fixed energy-level. The charge field accelerates in the direction of a lower potential so that the sum of the potential and kinetic energies remains constant.

11.7 Parallel magnetic Lorentz force term

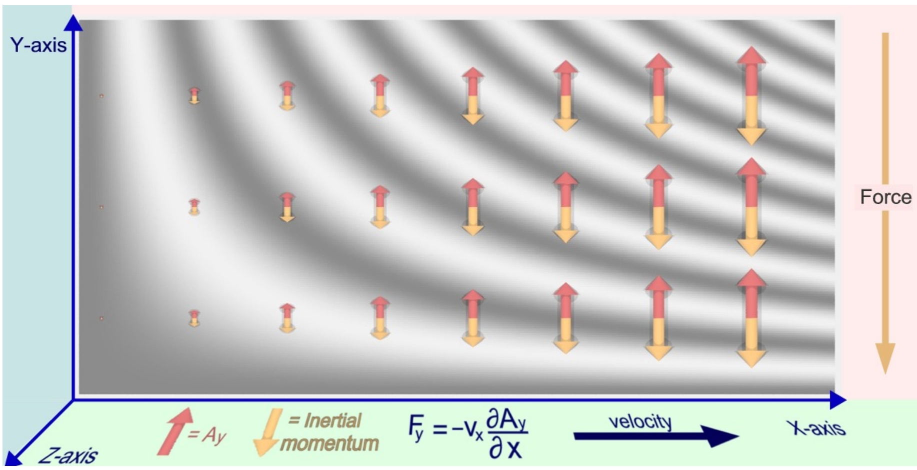

Figure 11.5: Magnetic force from the derivative of A_y in x

The parallel magnetic term $-v_x \partial_x A^y$ is shown above in figure (11.5). It is almost identical to the parallel electric term shown in figure (11.3). The difference is that the horizontal axis is now a *spatial* dimension, the x-axis, instead of the time dimension.

The particle is now assumed to have a velocity v_x along the x-axis. From the particle point of view, the horizontal axis can thus be seen as a time axis: the place where it we be after a certain amount of time. This is the reason for the correspondence between figure (11.5) and figure (11.3).

The vertical phase change rate is a direct result of the A_y component. Locality and special relativity require that the field ψ cancels this by accelerating in the opposite direction of A^y.

We see that the general rule is that: All space-like integrals of $eA^\mu\psi$ are inhibited by special relativity. Only time-like integrals (path-integrals) are allowed. This means that the vector field A^μ does *not* modify the phase ϕ of the field at different spatial locations directly. The influence of A^μ comes only over time because the accelerations change the inertial energy of the field and thus the phase change rate in time.

11.8 Orthogonal magnetic Lorentz force term

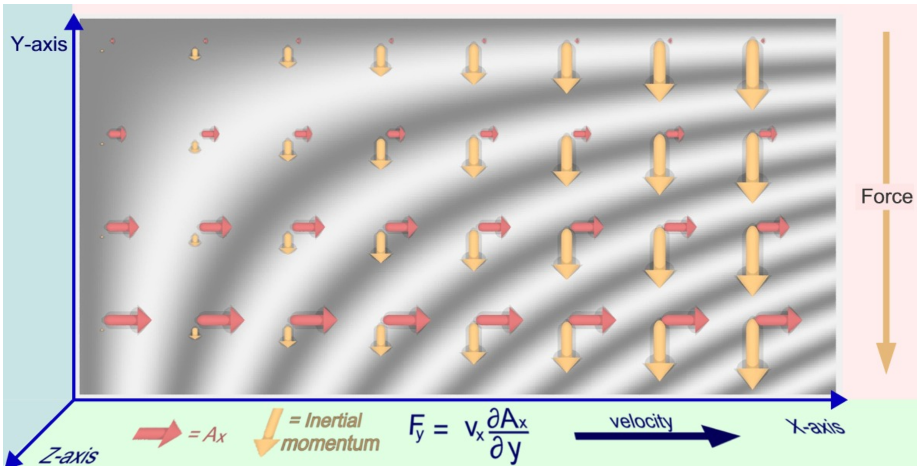

Figure 11.6: Magnetic force from the derivative of A_x in y

Finally, the orthogonal magnetic term $v_x \partial_y A^x$ is shown in figure (11.6). In this case a vertical phase change rate builds up as a result of the changing A^x, which increases from bottom to top.

The horizontal induced phase change rates are a *direct* result of A^x, while the vertical phase change rate is an *indirect* effect. This side effect occurs because the phase ϕ is a scalar value while A^μ is a vector.

A scalar value can not contain the same amount of information as a vector field so different vector fields A^μ inevitably must map to the same phase field ϕ. These different field configurations are of course the four cases which have been presented here.

The field ψ moving to the right has to cancel the indirectly induced phase change rate by accelerating downwards along the y-axis because of locality and special relativity.

The acceleration is in this case a *combined* effect of the scalar nature of the field plus the locality required by special relativity. The same is true in the case of the orthogonal electric case as shown in figure (11.4)

11.9 The total four-vector Lorentz force

For a fully relativistic theory we want all four components of the Lorentz force. To find the change in inertial energy $\partial p^o/\partial t$ we start by taking the time derivative of the usual relation below for the inertial momentum which stays valid even when there is interaction.

$$\frac{\partial}{\partial t}\Big\{\, p_o^2 \,\Big\} \quad = \quad \frac{\partial}{\partial t}\Big\{\; p_x^2 + p_y^2 + p_z^2 + m^2c^2 \;\Big\} \tag{11.42}$$

With the substitution of $\vec{v} = \vec{p}/p^o$ we can write this as.

$$\frac{\partial p^o}{\partial t} \quad = \quad \vec{v} \,\cdot\, \frac{\partial \vec{p}}{\partial t} \tag{11.43}$$

Substituting the Lorentz force in gives us.

$$\frac{\partial p^o}{\partial t} \quad = \quad \vec{v}\,\cdot\, e\Big(\mathsf{E} + \vec{v}\times \mathsf{B}\Big) \tag{11.44}$$

and since the last term containing the magnetic field is zero per definition we get for the change in inertial energy.

$$\frac{\partial p^o}{\partial t} \quad = \quad e\,\vec{v}\cdot \mathsf{E} \tag{11.45}$$

We did see here that the magnetic field can not increase or decrease the inertial energy density of the particle because the acceleration is orthogonal to the velocity. The inertial energy density is what we would infer from the speed, the particle's mass and the local density.

This result now allows us to express the total relativistic four-acceleration as follows in Lorentz Heaviside units (with c=1).

$$\frac{\partial U^\nu}{\partial\,\tau} \quad = \quad \frac{e}{m}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ -\beta_x\gamma \\ -\beta_y\gamma \\ -\beta_z\gamma \end{pmatrix} \quad = \quad \frac{e}{m}\,F^{\nu\mu}\,U_\mu \tag{11.46}$$

Where τ is the local proper time γt and where U^μ is the relativistic four velocity $\partial x^\mu / \partial \tau$.

We can split the four-vector Lorentz force in parallel and orthogonal components, where the parallel components of $\partial U^\nu / \partial \tau$ contain the four potential A^ν in the same direction.

$$\frac{\partial U^\nu}{\partial \tau} = \frac{e}{m}\Big(\partial^\mu A^\nu - \partial^\nu A^\mu \Big) U_\mu \tag{11.47}$$

So the first term at the right hand site contains the parallel components while the second term contains the orthogonal components.

The parallel components arise from special relativity. The phase induced by A^ν causes an acceleration in the direction (or opposite direction) of A^ν which cancels the phase. The phase must be canceled because the local phase is determined by a spatial integral which requires information from A and ψ from everywhere. Such an integral would require instantaneous communication if the effects on the phase don't cancel.

The orthogonal components are added to assure that the phase change, while going around a Wilson loop, is zero. The orthogonal components thus arise from the restriction that the total (canonical) momentum is determined by the phase change rates, where by the phase ϕ is a scalar. We can say that the orthogonal components arise from the $U(1)$ symmetry of the Klein Gordon field where $U(1)$ is the group containing all possible values of $\exp(i\phi)$.

The results obtained here can be carried over one-to-one to the Dirac field which, unlike the Klein Gordon field, has also a spin and corresponding magnetic moment besides a charge. One can decompose the charge-current density of the Dirac field into two components using the so-called Gordon decomposition. One of the components is exactly the same as that of the Klein Gordon field and it represents the charge of the electron.

The other component turns out to represent the charge-current density due to the electron's inherent magnetic moment caused by its spin. This spin based charge-current density produces an additional four-potential field A^μ_s and responds to other external potential fields.

11.10 Maxwell's equations

It is now easy to verify Maxwell's laws. We have derived the four vector Lorentz force and the corresponding Faraday tensor $F^{\mu\nu}$ containing the six components of the electro-magnetic field corresponding with the six possible Wilson loops in 4d space.

There is not really much to add here from a quantum field perspective besides a few remarks. All which is required is elementary classical electromagnetism, and this section is merely added for completeness.

Maxwell's inhomogeneous equations

Starting with the two inhomogeneous equations which yield the source of the field (the charge-current density).

We have Gauss' law for the electric field.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{11.48}$$

and Ampre's circuital law. (both laws are given in SI units).

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{11.49}$$

Maxwell's inhomogeneous equations can be expressed more elegantly in the relativistic four-vector form:

$$\left| \begin{array}{cccc} 0 & -\,\frac{1}{c}\partial_x E_x & -\,\frac{1}{c}\partial_y E_y & -\,\frac{1}{c}\partial_z E_z \\ +\,\frac{1}{c}\partial_o E_x & 0 & -\;\;\partial_y B_z & +\;\;\partial_z B_y \\ +\,\frac{1}{c}\partial_o E_y & +\;\;\partial_x B_z & 0 & -\;\;\partial_z B_x \\ +\,\frac{1}{c}\partial_o E_z & -\;\;\partial_x B_y & +\;\;\partial_y B_x & 0 \end{array} \right| \begin{array}{c} = \\ = \\ = \\ = \end{array} \left| \begin{array}{c} -\mu_o\, J_o \\ -\mu_o\, J_x \\ -\mu_o\, J_y \\ -\mu_o\, J_z \end{array} \right| \tag{11.50}$$

Which is an explicit way of writing (in SI) the well know expression.

$$\partial_v F^{\mu\nu} = -\mu_o J^\mu \tag{11.51}$$

One interesting observation from a quantum theory perspective can be made if we split $F^{\mu\nu}$ in its parallel and orthogonal components.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{11.52}$$

For Maxwell's inhomogeneous equations we get.

$$\partial^\mu \partial_v A^\nu - \partial^\nu \partial_v A^\mu = -\mu_o J^\mu \tag{11.53}$$

The first term is zero, it contains the expression for the conservation of the vector potential.

$$\partial_v A^\nu = 0 \tag{11.54}$$

The total net "current" $\vec{A}$ streaming out of a volume element dx^3 is equal the the decrease in time of A^o.

This term contains the orthogonal components which are due to the scalar nature of the field ψ. The orthogonal components are thus not related to the charge-current density J^μ.

The second term gives us the wave equation for the electromagnetic potential field.

$$\partial^\nu \partial_v A^\mu = \mu_o J^\mu \tag{11.55}$$

The parallel components of the Lorentz force are thus involved in the source of the electromagnetic field.

Maxwell's homogeneous equations

The first homogeneous equation is Gauss' law for magnetism

$$\nabla \cdot \mathbf{B} = 0 \tag{11.56}$$

Which follows from $\mathbf{B} = \nabla \times \vec{A}$, and we have Faraday's law of induction.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11.57}$$

Which follows from $\mathbf{E}{=}{-}\partial_t \vec{A} - \nabla\vec{\Phi}$ and $\partial_v A^\nu{=}0$, the conservation law.

Maxwell's homogeneous equations can be expressed more elegantly in relativistic four-vector form:

$$\left| \begin{array}{cccc} 0 & -\ \partial_x B_x & -\ \partial_y B_y & -\ \partial_z B_z \\ +\ \partial_o B_x & 0 & +\ \frac{1}{c}\partial_y E_z & -\ \frac{1}{c}\partial_z E_y \\ +\ \partial_o B_y & -\ \frac{1}{c}\partial_x E_z & 0 & +\ \frac{1}{c}\partial_z E_x \\ +\ \partial_o B_z & +\ \frac{1}{c}\partial_x E_y & -\ \frac{1}{c}\partial_y E_x & 0 \end{array} \right| \begin{array}{c} = \\ = \\ = \\ = \end{array} \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right| \tag{11.58}$$

Which is an explicit way of writing (in SI) the well know expression.

$$\partial_v \ {}^*F^{\mu\nu} \ = \ 0 \tag{11.59}$$

where ${}^*F^{\mu\nu}$ is the so called Hodge[1] dual of $F^{\mu\nu}$. Thus, ${}^*F^{\mu\nu}$ is a contravariant tensor which has the E's and B's swaped.

[1]Note that this popular use of differential geometry assumes a non relativistic 3d space where the E's are one-forms and de B's are two-forms. However, as we have seen from the Wilson loop treatment, both E and B are two forms defined by exterior products in 4d Minkowski space.

11.11 Covariant derivative and gauge invariance

Our starting point in this chapter was the interacting Klein Gordon equation. Here we already assumed how A^μ gives rise to phase changes. We can ask ourself how this relation between A^μ and the phase of ψ arises in the first place. This brings us to the subject of the so-called gauge transformations.

We did see that we can define the phase of the field as integrals over time and space.

$$\psi \;= \exp\left\{ -\frac{i}{\hbar}\int \Big(\; p^o + eA^o \;\Big)dx^o \;\; + \;\; \frac{i}{\hbar}\sum_{i=1}^{3}\int \Big(\; p^i + eA^i \;\Big)dx^i \right\} \quad (11.60)$$

Firstly this expression seems over determined, however the scalar nature of ϕ and special relativity just constrains the values of p^μ and A^μ. The expression relates the two in such a way that a change in A^μ must give rise to a corresponding change in p^μ, and this change in p^μ corresponds to the Lorentz force.

The spatial integrals are not allowed in special relativity since they would result in instantaneous information transport. This means that a variation δA^i must be compensated by a variation δp^i. The scalar nature of ϕ then introduces a corresponding set of terms which guarantees that the change in phase integrated over a Wilson loop is zero.

We can obtain extra insights about the constraints and degrees of freedom of A^μ and p^μ with the theory of *local gauge invariance*. The field ψ is said to be *global* gauge invariant because the physics doesn't change if we add a constant phase to it globally.

$$\psi \quad \Rightarrow \quad e^{i\alpha}\psi \qquad (11.61)$$

Where α is a global constant independent of place and time. This is evident, but what if we make this variation local by defining a function $\Lambda(x^\mu)$ which can vary from place to place?

$$\psi \quad \Rightarrow \quad e^{i\Lambda}\psi \qquad (11.62)$$

If we assume that Λ^μ is a variation on the phase due to A^μ then the above is how the field ψ transforms under the variation Λ. The variation of the A^μ itself due to Λ is given by.

$$A_\mu \quad \Rightarrow \quad A_\mu + \frac{1}{e}(\partial_\mu \Lambda) \tag{11.63}$$

Who will this variation Λ^μ of A^μ effect the inertial four-momentum p^μ? We can express p^μ in the interacting case as:

$$p^\mu \quad = \quad \frac{i\hbar c}{2m}\Big(\ \psi^* \overleftrightarrow{D^\mu} \psi\ \Big) \quad = \quad \frac{i\hbar c}{2m}\Big(\ \psi^* D^\mu \psi \quad + \quad \psi\ D^{\mu *} \psi^*\ \Big) \tag{11.64}$$

$$\text{where:} \qquad D^\mu \quad = \quad \partial^\mu - ieA^\mu$$

From equation (11.63) we see that the variation D^μ becomes.

$$D^\mu \quad \Rightarrow \quad \partial^\mu - ieA^\mu - i(\partial^\mu \Lambda) \tag{11.65}$$

From now on we will use the words "transforms as", where the transform is said to be a *gauge* transformation. So, from the above we can now work out that $D^\mu\psi$ transforms as

$$D^\mu \psi \quad \Rightarrow \quad e^{i\Lambda}\, D^\mu \psi \tag{11.66}$$

and the complex conjugate ψ^* transforms as.

$$\psi^* \quad \Rightarrow \quad \psi^*\, e^{-i\Lambda} \tag{11.67}$$

Combining the two expressions above we see that.

$$\psi^* D^\mu \psi \quad \Rightarrow \quad \psi^* D^\mu \psi \tag{11.68}$$

This term is thus *invariant* under a variation of the phase with Λ and this means that the four-momentum density p^μ is also invariant under such a phase variation with a scalar field Λ.

$$p^\mu \quad \Rightarrow \quad p^\mu \tag{11.69}$$

This is not so surprising since a scalar phase field has the property that the change in phase integrated over a Wilson loop is zero. The begin and end points are the same and they obviously have the same value.

We can therefor say that expressions like $\psi^* D^\mu \psi$, $\psi^* D_\mu \psi$ and p^μ are gauge invariant expressions. This then validates the name *gauge covariant derivative* for an expression like $D_\mu = \partial_\mu - ieA_\mu$. If we would have used the ordinary derivative then the resulting momentum density would not have been gauge invariant.