The simplest derivation of the non Abelian field tensor and Lagrangian density.

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First the Abelian electromagnetic field is handled after which the same steps are repeated for the non Abelian case. The Field tensors are derived from the operator that determines the phase change rates when acting on the fields. The starting point of the Lagrangian density is simply the scalar Lagrangian which is extended to the more complex groups by taking symmetries in account which lead to the addition of terms which produce indistinguishable phase change rates in U(1) and SU(2) respectively.

I. THE ABELIAN FIELD TENSOR

In the Abelian case we have a scalar field ψ with an U(1) symmetry. Such a field with inertial momentum p^μ including the momentum from electromagnetic interaction A^μ is given by:

$$\psi \;=\; \psi^p\psi^A \;=\; \exp\left\{-\frac{i}{\hbar}x_\mu\Big(\;p^\mu + eA^\mu\;\Big)\right\} \qquad (1)$$

If the inertial momentum p^μ, and the electromagnetic momentum eA^μ are not constant then we need to take the the integrals along the coordinate axis.

$$\psi \;=\; \psi^p\psi^A \;=\; \exp\left\{-\frac{i}{\hbar}\int dx_\mu\Big(\;p^\mu + eA^\mu\;\Big)\right\} \qquad (2)$$

This is an over specified expression. The phase is a scalar while p^μ and A^μ are four vectors. This restricts the possible values of p^μ and A^μ. More specifically, the four-curl of the total momentum $p^\mu + eA^\mu$ must be zero. The integral is the inverse of the (four)gradient operator and the curl of a gradient is zero per definition.

Changes in A^μ require a cancelation by opposite changes in p^μ to keep the integral a simply connected scalar field. These changes of p^μ of course represent the Lorentz force. The operator which gives the changes of the four momentum when acting on ψ is given by.

$$\partial_\mu\, p_\nu \psi^p \;=\; i\hbar\,\partial_\mu\partial_\nu\psi^p \qquad (3)$$

Which can be seen by applying this operator on (2). The order of the $\partial_\mu\partial_\nu$ derivatives should make no difference in the case of the complete field ψ and therefor.

$$i\hbar\Big(\;\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\;\Big)\psi \;=\; 0 \qquad (4)$$

This states that the curl of $p^\mu + eA^\mu$ is zero. However, if we separate ψ into p^μ and A^μ factors then we have the individual fields ψ^p and ψ^A which are not simply connected anymore on their own.

$$F^{\mu\nu}\,\psi^A = \frac{i\hbar}{e}\Big(\;\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\;\Big)\psi^A \qquad (5)$$

If we let the differential operators act on the on the "A^μ only" part of (2) then we obtain.

$$F^{\mu\nu}\,\psi^A = \Big(\;\partial^\mu A^\nu - \partial^\nu A^\mu\;\Big)\psi^A \qquad (6)$$

We have derived the Abelian field tensor corresponding to the electro magnetic field.

II. THE ABELIAN LAGRANGIAN

Although the Lagrangian density of the electromagnetic field looks rather different compared to the scalar Lagrangian density. It's actually just a few steps away from the Klein Gordon Lagrangian density which is given by.

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \qquad (7)$$

We discard the mass term and we replace the scalar field ψ with the four components of the vector field A^μ coosing the indices so that the resulting Lagrangian density transforms like a Lorentz scalar. The result is, up to a unit system dependent constant:

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A_\nu) \qquad (8)$$

However, we have to replace all terms like $\partial_\mu A_\nu$ with $(\partial_\mu A_\nu - \partial_\nu A_\mu)$. The phase which the two terms between brackets induce in a scalar field is indistinguishable. We therefor get.

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A_\nu - \partial^\nu A_\mu) \qquad (9)$$

So only the curl of the potential A^μ contributes to the Lagrangian density of the electro magnetic field. Consequently it can be expressed in terms of the field tensor.

$$\mathcal{L} \;=\; -\frac{1}{4\mu_o}F^{\mu\nu}F_{\mu\nu} \;=\; \frac{1}{2}\Big(B\cdot H - D\cdot E\Big) \qquad (10)$$

Normalized in SI units.

III. THE NON-ABELIAN FIELD TENSOR

We can derive the non-Abelian field tensor in exactly the same way. The non-Abelian interaction is given by.

$$\psi^G \quad = \quad \exp\left\{-\frac{ig}{\hbar}T^i\int dx_\mu G_i^\mu\right\} \tag{11}$$

Where the T^i are the generators of the group. In case of the SU(2) group these are the Pauli-matrices divided by two. We can now derive the non-Abelian field tensor in exactly the same way as in (5)

$$F^{\mu\nu}\,\psi^G \quad = \quad -\frac{i\hbar}{g}\left(\,\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\,\right)\psi^G \quad \neq \quad 0 \tag{12}$$

Showing the i-indices explicitly with the definitions $F^{\mu\nu} = T^i F_i^{\mu\nu}$ and $G^\mu = T^i G_i^\mu$ in mind, we derive.

$$\begin{aligned} T^i\,F_i^{\mu\nu} \;=\; & T^i\Big(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu\Big) - \\ & i\tfrac{g}{\hbar}\Big(T^j G_j^\nu\; T^k G_k^\mu \;-\; T^k G_k^\nu\; T^j G_j^\mu\Big) \end{aligned} \tag{13}$$

The last two terms, which cancel in the U(1) case, can be written as a commutation over the i-indices.

$$F^{\mu\nu} \quad = \quad \partial^\mu G^\nu - \partial^\nu G^\mu - ig\Big[G^\mu\;,\;G^\nu\Big] \tag{14}$$

This is the usual presentation. We have set $\hbar$ to 1 here. We can also group the terms of the non Abelian field tensor depending on from which term of $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\psi^G$ they originate.

$$T^i\,F_i^{\mu\nu} = \Big(\partial^\mu - igT^jG_j^\mu\Big)T^kG_k^\nu - \Big(\partial^\nu - igT^kG_k^\nu\Big)T^jG_j^\mu \tag{15}$$

Making the i-indices implicit and defining a gauge covariant derivative $D^\mu = \partial^\mu - igG^\mu$ we can simplify this to.

$$F^{\mu\nu} \quad = \quad D^\mu G^\nu - D^\nu G^\mu \tag{16}$$

This shows that the non Abelian field interacts with itself.

IV. THE NON-ABELIAN LAGRANGIAN

Going back to the scalar Lagrangian density.

$$\mathcal{L} \quad = \quad \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{17}$$

Applying this to the non Abelian field G^μ we use the "covariant derivative" D_μ since this is the expression which determines the phase change rates in this case: The rotation rates in SU(2). Ignoring the mass term we get

$$\mathcal{L} \quad = \quad tr\;\frac{1}{2}(D_\mu G_\nu)(D^\mu G_\nu) \tag{18}$$

The trace sums over the i-indices. We must however replace terms like $D_\mu G_\nu$ with ones like $(D_\mu G_\nu - D_\nu G_\mu)$ since the two terms between brackets contribute equally to the rotation rates in SU(2) and are there for indistinguishable. Hence the Lagrangian becomes.

$$\mathcal{L} \quad = \quad tr\frac{1}{2}(D_\mu G_\nu - D_\nu G_\mu)(D^\mu G_\nu - D^\nu G_\mu) \tag{19}$$

Which we express in the field tensor as follows.

$$\mathcal{L} \quad = \quad -\,tr\;\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{20}$$

Where the factor 1/4 stems from the structure coefficients.