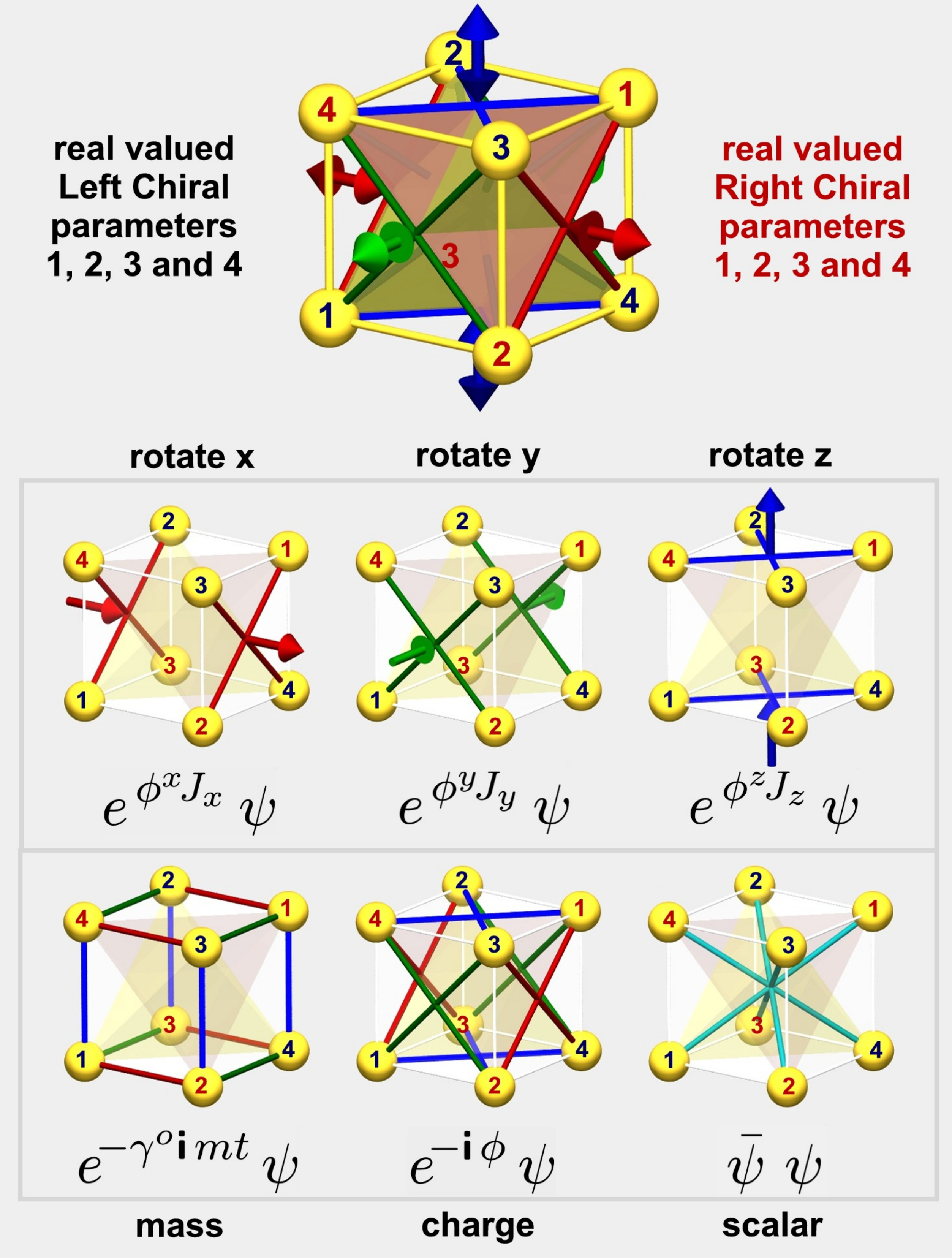
REAL VALUED, SPATIAL SYMMETRIC REPRESENTATION OF THE DIRAC EQUATION



Real valued, Spatial symmetric representation of the Dirac equation

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abstract

We describe how one can transform the standard representation of the Dirac equation into a new, spatially symmetric, real valued representation. It is shown how this new representation can be used as a direct replacement in QED and other standard model applications, it produces the same results and the notation is carefully developed to be virtually identical to the standard notation.

The real symmetric representation allows a simple geometric visualization of the eight parameters of the bi-spinor. The eight corresponding vectors $\bar{\psi}_e\gamma^\mu\psi_e$ symmetrically point to the eight vertices of a cube from the center outwards. Locating the parameters on this cube allows us to visualize the linear relations imposed by the gamma matrices acting on the bi-spinor. These visualizations turn out to be simple and easy interpretable, in contrast with the seemingly intricate algebraic operations of the complex gamma matrices.

The real symmetric representation replaces the 2×2 complex matrices by 4×4 real matrices and so $\mathrm{SO}(4) \cong \mathrm{Spin}(3) \otimes \mathrm{Spin}(3)$ becomes the representation's group of unitary generators. SO(4) contains all 6 possible rotation and phase generators while only 4 of these are used in the SU(2) representation (5 if Majorana particles are included). It is SO(4) which allows us to make the representation symmetric in x, y and z.

It will become apparent throughout this document that, as the result of SO(4), an extended Dirac equation arises which connects much better with the rest of the Standard Model physics beyond QED.

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1 Overview of the real symmetric representation

Chapter 1

Real Symmetric Representation

1.1 Motivation for a real valued representation

Single spinors with 2 complex numbers contain four independent parameters. Only the 4×4 real valued matrix operators with 16 coefficients are able to define 100% of all the possible linear relations between the four components of a spinor field. The 2 × 2 complex matrices used in the standard representation for instance are limited to just eight coefficients.

Algebra	$\mathbb{R}$ real	$\mathbb{C}$ complex	$\mathbb{H}$ quaternions
general group	$\mathrm{Mat}(4,\mathbb{R})$	$\mathrm{Mat}(2,\mathbb{C})$	$\mathbb{H}^{\mathrm{L}} \otimes \mathbb{H}^{\mathrm{R}}$
operator form	$\begin{pmatrix} \mathbb{R}\,\mathbb{R}\,\mathbb{R}\,\mathbb{R} \\ \mathbb{R}\,\mathbb{R}\,\mathbb{R}\,\mathbb{R} \\ \mathbb{R}\,\mathbb{R}\,\mathbb{R}\,\mathbb{R} \\ \mathbb{R}\,\mathbb{R}\,\mathbb{R}\,\mathbb{R} \end{pmatrix} \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \\ \mathbb{R} \\ \mathbb{R} \end{pmatrix}$	$\begin{pmatrix} \mathbb{C}\,\mathbb{C} \\ \mathbb{C}\,\mathbb{C} \end{pmatrix} \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$	$(\ \mathbb{H}^{\mathrm{L}}\ \mathbb{H}^{\mathrm{R}}\ )\,(\ \mathbb{H}\ )$
operator size	16 parameters	8 parameters	8 parameters

The 4×4 real matrix representation and thus the SO(4) group of unitary generators provides the most general representation. All other representations, not only the standard one, but also the quaternion based representations as well as David Hestenes' Real Dirac theory can be directly formulated in the 4×4 real matrix representation. This therefor provides a single framework to understand all other representations and the ways in which these representations interrelate.

Missing coefficients (degrees of freedom) in the matrix operators means a risk of missing essential physics. For instance, the extra degrees in freedom is what allows us to make the representation symmetric in x, y and z. In order to restore the missing coefficients we follow a two-step process. First we transform the complex into a real valued representation and secondly we remove the artificial restrictions imposed on the coefficients.

$$a + ib \xrightarrow[\hspace{3em}]{\text{complex to real}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \xrightarrow{\text{add coefficients}} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{1}$$

1.2 The three charge generators of SO(4)

A 4×4 real matrix representation implies the SO(4) group of unitary generators and its six unitary generators. This group decomposes in two subgroups like SO(4) ≅ Spin(3) ⊗ Spin(3). One of these two subgroups always becomes the group of absolute rotation generators in any representation, the real, the complex and the quaternion representation.

Algebra	$\mathbb{R}$ real	$\mathbb{C}$ complex	$\mathbb{H}$ quaternions
rotation group	Spin(3)⊗Spin(3)	Spin(3)⊗Spin(3)	Spin(3)⊗Spin(3)
space rotation	3d rotation $\mathbf{i}_i\sigma^i$	3d rotation $i\sigma^i$	3d rotation $\hat{q}^{\mathrm{R}}$
phase rotation	3 charges $\mathbf{i}_i$	3 charges $i,\ \sigma^2*,\ i\sigma^2*$	3 charges $\hat{q}^{\mathrm{L}}$

The second Spin(3) group always becomes a group of three orthogonal generators of charge, also in all representations. In the quaternion representation the two Spin(3) groups are for instance given by the left- and right-multiplying quaternions $\hat{q}^L$ and $\hat{q}^R$.

The Spin(3) group of charge generators also exist in the standard Dirac representation! but the three generators are expressed in a rather awkward way by $i\sigma^o$, σ^2* and $i\sigma^2*$. The first one, generally written as i, is just the generator of electric charge. The two other generators can not be expressed by 2×2 complex matrices due to the limitations mentioned but require a complex conjugate operator '*' (without transpose) acting on the spinor to the right.

The last generator is often mentioned as the charge conjugate operator and frequently used for the construction of Majorana particles. It is quite remarkable that the elementary fact that i, σ^2* and $i\sigma^2*$ have the operator commutation relations of a Spin(3) group is absent from the literature. This group has an important role to play the Standard Model physics beyond QED and is thoroughly studied in this document.

1.3 Basic elements of the real valued representation

The notation of the real representation is virtually identical to the standard notation and it is often only when we write out the Pauli matrices explicitly that the difference in notation becomes apparent. We can continue to use the standard notation because we reintroduce i. It is defined as the real valued 4×4 matrix being the formal replacement for the matrix $i\sigma^o$. This matrix is used in the real valued representation instead of the value i and it allows us to reuse most of the standard complex notation.

$$\mathsf{i}^2 = -\mathbf{1} \qquad \mathsf{i}^{-1} = -\mathsf{i} \qquad e^{\pm \mathsf{i}\phi} = \mathbf{1}\cos\phi \ \pm \ \mathsf{i}\sin\phi \tag{2}$$

We can further use $\mathsf{i}$in combination with the real valued 8×8 gamma matrices where it is applied as $\left(\begin{smallmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{i} \end{smallmatrix}\right)$. The Dirac equation in the new representation looks just like the standard one.

The Dirac equation

$$\begin{array}{lrcl} \text{standard notation:} & \gamma^\mu\left(\partial_\mu + ieA_\mu\right)\psi & = & -\,im\psi \\ & & \downarrow & \\ \text{symmetric and real:} & \gamma^\mu\left(\partial_\mu + \mathsf{i}eA_\mu\right)\psi & = & -\,\mathsf{i}m\psi \end{array} \tag{3}$$

Where the only difference in notation is that i has become the matrix i.

The 8 independent parameters of the bi-spinor field (4 real and 4 imaginary) become 8 real valued variables in the new representation. Each chiral component is represented by a 4×1 matrix. The Pauli matrices become 4×4 real matrices instead of 2×2 complex ones, doubling the number of elements per matrix. So, for instance, the complex

$$\psi_L^\dagger\,\sigma\,\psi_L \quad = \quad \begin{pmatrix} a+ib \\ c+id \end{pmatrix}^\dagger \begin{pmatrix} a_{00}+i\,b_{00} & a_{01}+i\,b_{01} \\ a_{10}+i\,b_{10} & a_{11}+i\,b_{11} \end{pmatrix} \begin{pmatrix} a+ib \\ c+id \end{pmatrix} \tag{4}$$

Becomes the entirely real valued expression.

$$\psi_L^\mathsf{T}\,\sigma\,\psi_L \quad = \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}^\mathsf{T} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{5}$$

The conjugate transpose † is replaced by T, the real transpose. We may replace † with T because the method we use to go from complex to real numbers follows the rule.

$$(a+ib)^{\dagger} \rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{\mathsf{T}} \tag{6}$$

You might want to check this with $\sigma_2^{\dagger} = \sigma_2$. There is yet another rule which allows us to express the spinors by 4×1 column vectors instead of the 4×2 matrices one would expect since we have to replace complex numbers by 2×2 matrices. If you look at the second column of the 2×2 matrix then you see that it can be obtained from the first column by.

$$\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

The matrix here is just the number i. Both columns contain the same information. The same happens in the real valued representation where the second column is given by.

$$\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \\ \psi_4' \end{pmatrix} = \mathbf{i} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{8}$$

The second column is just the first column multiplied by $\mathbf{i}$. This is what allows us to remove the second column from the representation. In the more general case an expression like $\psi_1^{\dagger}\psi_2$ will produce a complex result with both a real and an imaginary part. In the real valued representation these two are obtained independently by.

$$\begin{array}{ccc} \mathrm{re}(\ \psi_1^{\dagger}\psi_2\ ) & & \mathrm{im}(\ \psi_1^{\dagger}\psi_2\ ) \\ \downarrow & & \downarrow \\ \psi_1^{\mathsf{T}}\psi_2 & & -\ \psi_1^{\mathsf{T}}\,\mathbf{i}\,\psi_2 \end{array} \tag{9}$$

Contractions such as this one tend to reintroduce complex notation because they produce two results. In practice we will reuse the standard notation using 4×1 spinors and explicitly write out some elementary 'real' and 'imaginary' algebra while actually using only real matrices, when necessary.

1.4 From SU(2) to SO(4) spinor operator representation

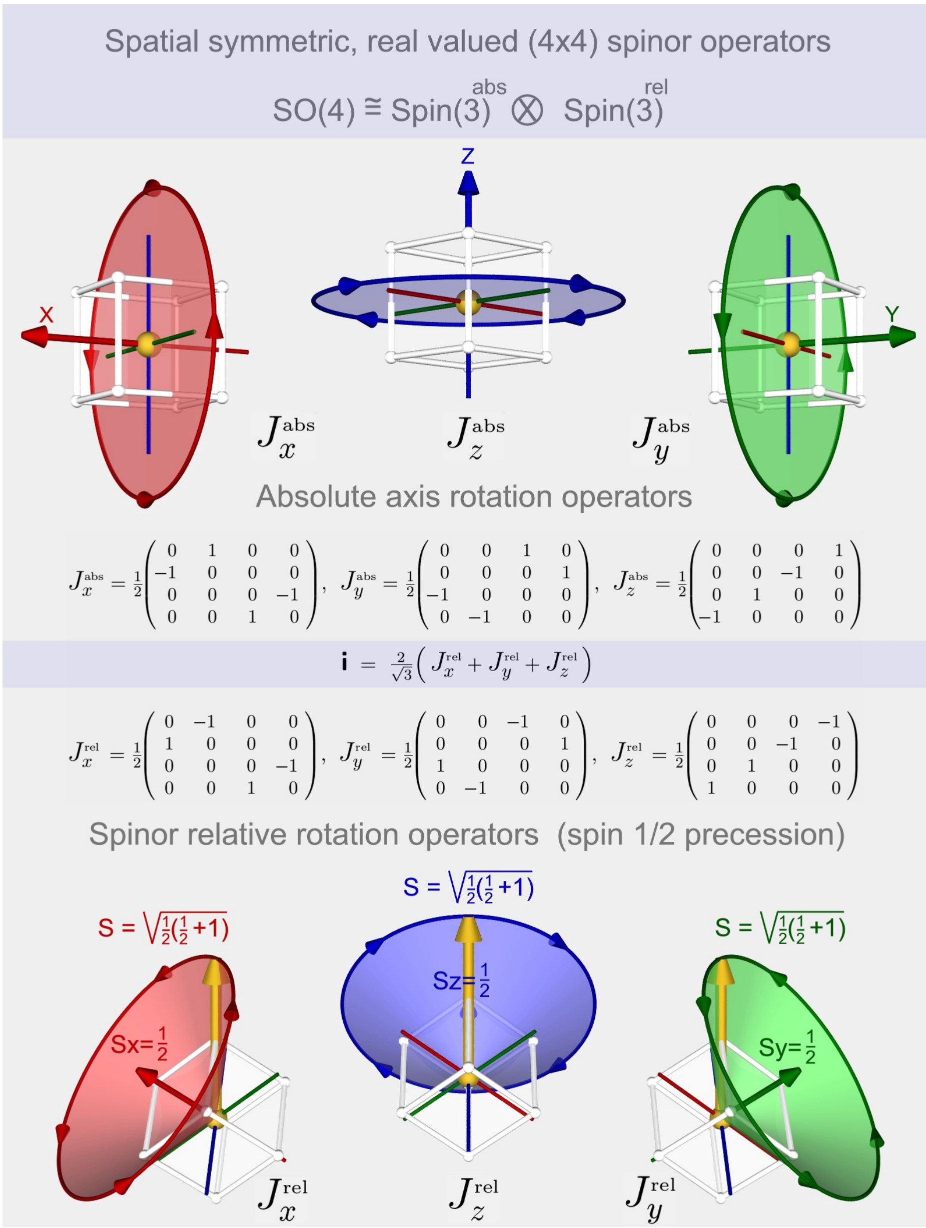

Figure 1: Under the symmetric choice for the charge generator $\mathbf{i}$ the three J_i^{L} generate relative spinor precession. If added together they act as $\mathbf{i}$

SU(2) spinor operator representation

In the complex SU(2) spinor representation there are four generators of rotation based on the four matrices $i\sigma^\mu$. The first one $i\sigma^o$ is the U(1) generator of phase and is typically written as just i. The other three are the rotators in 3 dimensional space.

The U(1) rotator has the beautiful fundamentally important property that it always rotates a spinor around its *own axis* entirely independent of the direction of the spinor. For a spinor ξ_s pointing in the $\vec{s}$ direction we can therefor write with $(i\vec{s}\cdot\vec{\sigma})$ as the rotation generator.

$$e^{is}\,\xi_s \;\;=\;\; e^{i\vec{s}\cdot\vec{\sigma}}\,\xi_s \tag{10}$$

The reason for this is that we define the spinor ξ_s as being an eigenvector of the boost matrix $(\vec{s}\cdot\vec{\sigma})$ in the $\vec{s}$ direction, and we therefor may replace the $(\vec{s}\cdot\vec{\sigma})$ at the right hand side by the eigenvalue s at the left hand side.

The U(1) operator e^{is} is thus a *relative* rotator, depending only on the orientation of the spinor, while the other three are *absolute* rotators fixed to the coordinate system. Important is that a field based on U(1) leads to rotational independent physics.

SO(4) spinor operator representation

In the real valued SO(4) spinor representation something even more beautiful happens. The six rotators of SO(4) decompose into two sets of 3 rotators in three dimensional space. One set again rotates absolute around the coordinate system axis but the other set of 3 generators rotates relative to the spinor orientation like the U(1) operator. Group theoretically:

$$\mathrm{SO}(4) \;\cong\; \mathrm{Spin}(3)^{abs} \otimes\; \mathrm{Spin}(3)^{rel} \tag{11}$$

This relative rotation is possible because a spinor has, besides a direction, also an orientation around its own axis given by its phase. A spinor is often depicted as a flagpole with the pole giving the direction and the flag giving the phase. A spinor can therefor span a coordinate system and it is in this relative coordinate system in which the $\mathrm{Spin}(3)^{rel}$ group operates. A field based on $\mathrm{Spin}(3)^{rel}$ leads to rotational independent physics.

The two sets of rotation generators, called $J_i^{\rm abs}$ and $J_i^{\rm rel}$, commute with each other. The first one is the normal rotation generator J_i and we use the latter for the replacement of i using $\frac{1}{2}\mathbf{i}_i = J_i^{\rm rel}$ and so we may write.

$$\begin{array}{lrcl} \text{SU(2) representation :} & [\; J_i \;,\; i \;] & = & 0 \\ \text{SO(4) representation :} & [\; J_i \;,\; \mathbf{i}_j \;] & = & 0 \end{array} \qquad (12)$$

This means that we can use any linear combination of $\mathbf{i}_i$ to define a generator $\mathbf{i}$ which is equivalent to the U(1) generator i in the SU(2) representation. This generator will be Abelian due to the above commutation rules.

$$\mathbf{i} \;=\; c_x\mathbf{i}_x + c_y\mathbf{i}_y + c_z\mathbf{i}_z \qquad (13)$$

The group formed by the $\mathbf{i}_i$ generators rotates relative to the spinor itself. Like in the SU(2) representation we define the spinor ξ_s pointing in the $\vec{s}$ direction to be an eigenvector of the boost matrix $\vec{s}\cdot\vec{\sigma}$ in the $\vec{s}$ direction and therefor the generator matrix $\mathbf{i}$ rotates the spinor around its own axis:

$$e^{\mathbf{i}\vec{s}\cdot\vec{\sigma}}\,\xi_s \;=\; e^{\mathbf{i}s}\,\xi_s \qquad (14)$$

Using the original SO(4) generators we find that the ξ_s must be the eigenvectors of the product of the two Spin(3) groups: $c_iJ_i^{\rm rel}s_jJ_j^{\rm abs}$ and we derive a simple expression which gives us the four real values of all spinors ξ_s at arbitrary phase ϕ, spin direction s_i and base c_i.

Definition of the real valued spinors

$$\xi_s \;=\; \frac{(\; s + c \;)}{\|\; s + c \;\|}\; \mathrm{e}^{-\mathbf{i}\,\phi} \qquad (15)$$

$$\begin{array}{lcl} s & = & (\,0,\; s_x,\; s_y,\; s_z\,) \\ c & = & (\,0,\; c_x,\; c_y,\; c_z\,) \end{array} \qquad (16)$$

The down spinors are simply those with the base $(-c_i)$. Note that we can use the s_x, s_y and s_z values directly to define the spinor parameters without doing any calculations! This truly beautiful feature is obscured in the SU(2) presentation. The unit vector r is per definition non zero and it therefor has to shift one or more coordinates. In the SU(2) case it shifts the z value of the spin. The Riemann sphere projection geometry is just the result of the normalization factor in the denominator.

Different choices of i lead to different base vectors.

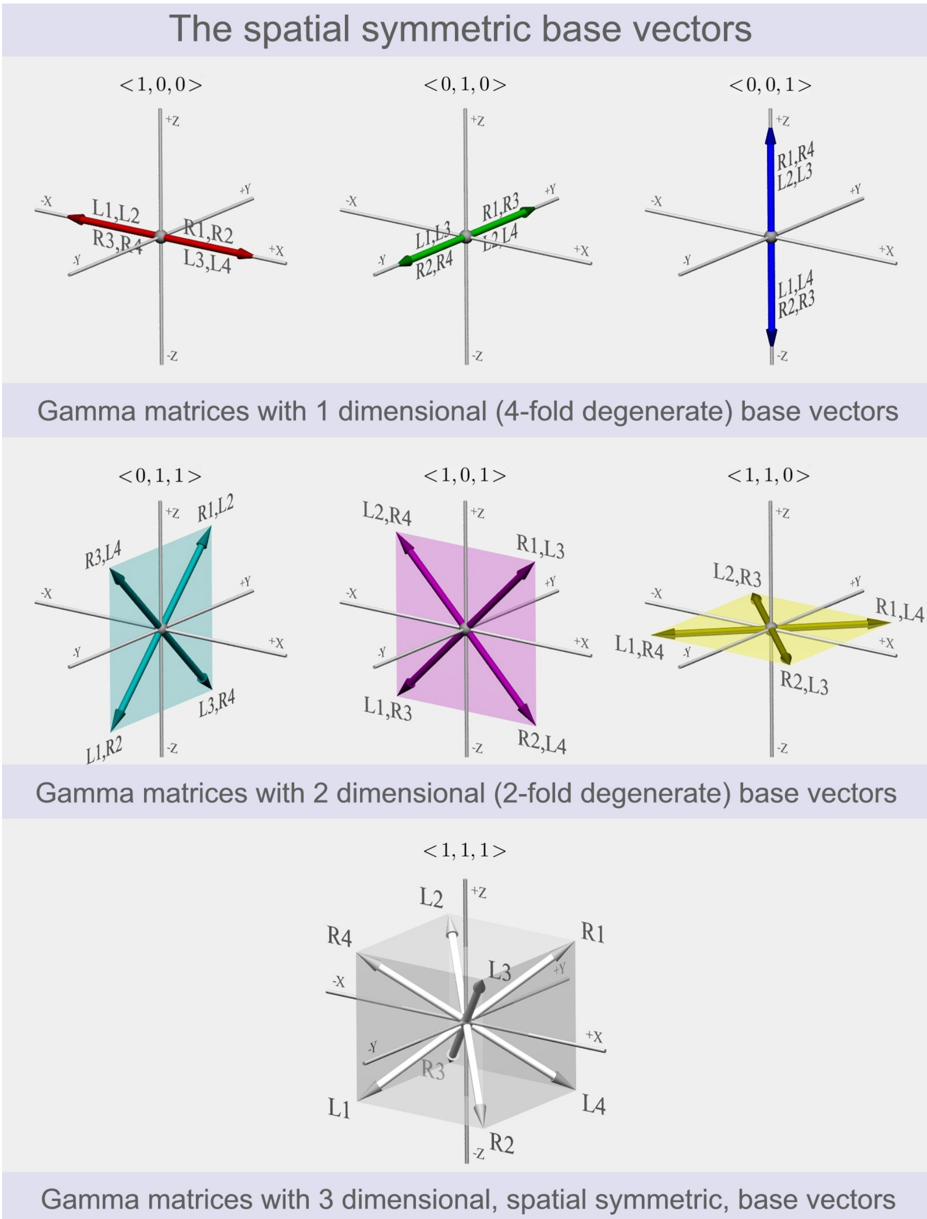

Figure 2: The left handed ($L_1..L_4$) and right handed ($R_1..R_4$) base-vectors as determined by particular choices of the generator of charge $\mathbf{i} = c_i \mathbf{i}_i$

1.5 Making the base vectors symmetric in 3 dimensions

We now have a representation which allows use to use any direction instead of the fixed z-axis in the SU(2) representation. It allows us in some sense to "look" at the representation itself from any angle. From figure (2) it shows that the representation can actually be viewed as a cube with the 8 independent bi-spinor parameters on the vertices.

If we look at a cube in normal space then under certain directions we see only a 2 dimensional square. We can label these specific directions the x, y and z axis. Looking from these directions we might miss the fact that we are actually looking at a cube.

In the spinor representation this effect is even more profound. If we "look" at the representation from the x, y and z directions then all we "see" is just a 1 dimensional line along that axis and we miss out completely on the fact that we are "looking" at a cube.

If we want to treat the three directons x, y and z on equal footing in our representation then there is only one direction which does this, the direction which fixed our definition of $\mathbf{i}$ as.

$$\mathbf{i} \;=\; \tfrac{1}{\sqrt{3}}\left(\,\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z\,\right) \tag{17}$$

Figure (2) shows us the base-vectors of various representations. The representation becomes spatially symmetric in three dimensional space if we make the *base vectors* symmetric. The bi-spinor has eight of these base vectors corresponding to the eight parameters it has. The base vectors are defined as $\bar{\psi}_e\,\gamma^\mu\,\psi_e$ where the bi-spinor of each base vector has only one non-zero parameter equal to 1.

The bi-spinor in the SU(2) representation has four real and four imaginary parameters. The spatial part of the eight base vectors has only z-components because only σ_z has non-zero elements on the main diagonal which maps the 1's to the 1's. The base vectors are four-fold degenerate and spatially oriented in a one dimensional symmetric way as shown in the upper part of figure (2). We obtain the real valued Pauli matrices by replacing each complex value in the SU(2) representation with a real matrix $a + ib \rightarrow \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Then we get the SO(4) representation if we allow any two-by-two real matrix. In the SO(4) representation then these extra degrees of freedom allow us to define three independent sets of Pauli matrices instead of one. These define the gamma matrix sets γ_1^μ, γ_2^μ and γ_3^μ, one for each axis, with corresponding base vectors.

$$\bar{\psi}_e\,\gamma_1^\mu\,\psi_e, \qquad \bar{\psi}_e\,\gamma_2^\mu\,\psi_e, \qquad \bar{\psi}_e\,\gamma_3^\mu\,\psi_e \tag{18}$$

This is the definition of the base-vector sets as shown in the upper part of figure (2). Each linear combination of these matrix sets can be equally well used for the gamma matrices. By adding the matrix sets together we add the resulting base vectors together. So, for instance we can define three sets of two-fold degenerate base vectors which are oriented symmetrically in a two-dimensional plane as shown in the middle of figure (2)

$$\bar{\psi}_e\tfrac{1}{\sqrt{2}}(\ \gamma_2^\mu+\gamma_3^\mu\ )\,\psi_e, \qquad \bar{\psi}_e\tfrac{1}{\sqrt{2}}(\ \gamma_3^\mu+\gamma_1^\mu\ )\,\psi_e, \qquad \bar{\psi}_e\tfrac{1}{\sqrt{2}}(\ \gamma_1^\mu+\gamma_2^\mu\ )\,\psi_e \tag{19}$$

We achieve our goal of a three dimensional spatial symmetric representation by simply adding the three sets of gamma matrices together with the correct scale factor. The symbols x, y and z are treated on equal footing and the representation becomes maximally symmetric. The eight corresponding base vectors point to the eight vertices of the unit cube.

$$\bar{\psi}_e\,\gamma^\mu\,\psi_e \;=\; \bar{\psi}_e\tfrac{1}{\sqrt{3}}(\ \gamma_1^\mu+\gamma_2^\mu+\gamma_3^\mu\ )\,\psi_e \tag{20}$$

The rotation group $\mathbf{i}_i$ is continuous but the eight real parameters of the bi-spinor can be considered to be part of a discrete group, for instance this cube which we will use to visualize the operators acting on the bi-spinor.

If we limit ourself to QED then we do not need to look at the $\mathbf{i}_i$ anymore and just use the $\mathbf{i}$ generator. Going beyond we see that all three $\mathbf{i}_i$ precess any spinor with exactly the angle required for a spin 1/2 particle as determined by the ratio between s_z and s. This angle corresponds with the angle of the $<1,1,1>$ direction with the $<1,0,0>$, $<0,1,0>$ and $<0,0,1>$ directions. In the SO(4) representation we can split the vector current into three individual precessing currents as in (20). Individually these precessing vector currents would emit high frequency radiation but combined together they act as $\mathbf{i}$ and the radiating parts cancel. In figure (1) one can see from the projections of S_x, S_y and S_z on the (vertical) spinor that each generator contributes 1/3 to the total Abelian generator of charge.

1.6 Overview of the real bi-spinor parameter visualization

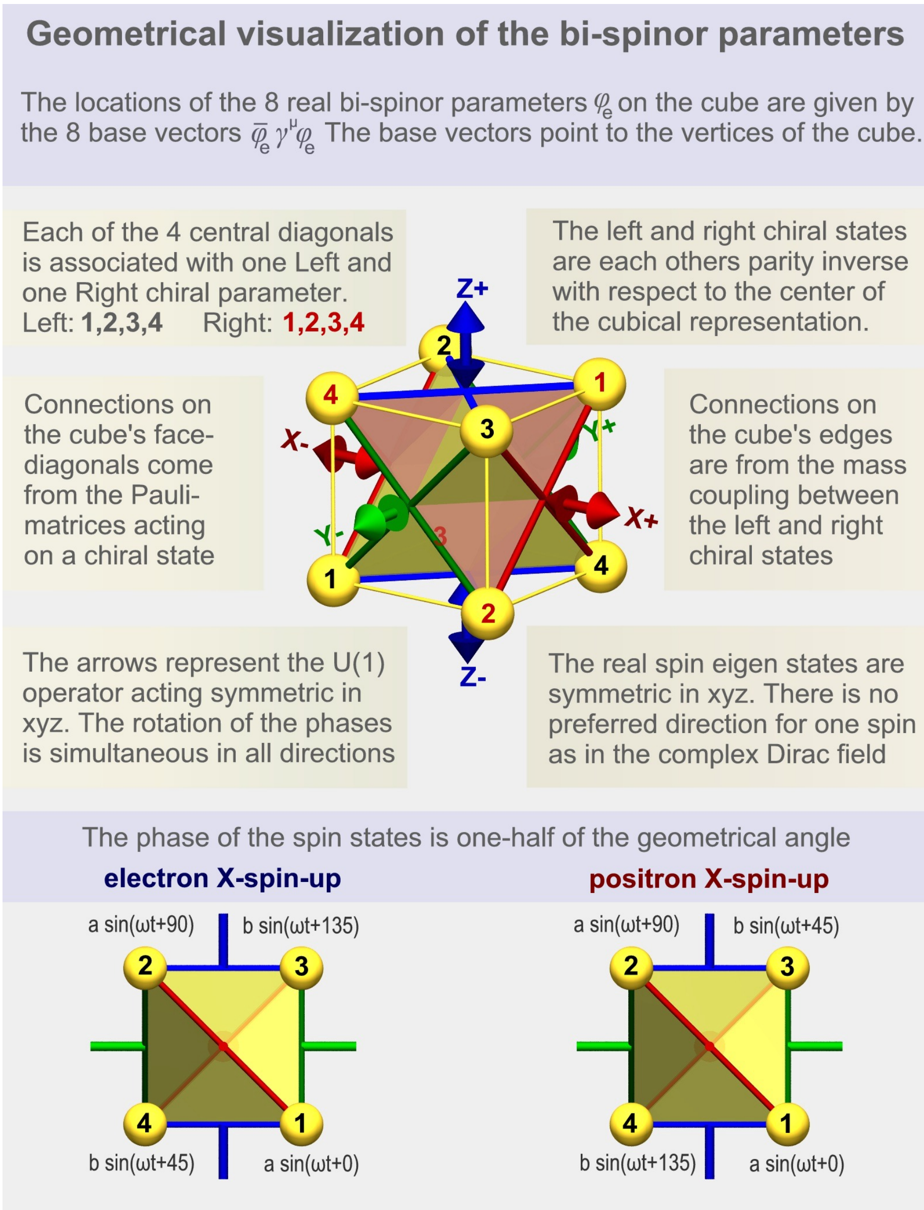

Figure 3: Visualization of the interpretation and key points

Very useful is a geometrical visualization of the eight values of the bi-spinor fermion field. Since the eight corresponding base-vectors $\bar{\psi}_e \, \gamma^\mu \, \psi_e$ point to the vertices of a cube we can visualize the values as being localized on the eight vertices of an idealized (infinitely small) cube aligned to the *xyz* coordinate system.

The U(1) phase: The vertices are the end points of the four central axis of the cube. Each axis is associated with a left and a righthanded base-vector. All the base-vectors transform light-like. The U(1) group occurs because vectors and axial vectors are defined with four axial orientations. There is an infinite number of ways in which a vector can be defined in this way because of the one parameter redundancy: the phase of the field. This encodes the U(1) group in a symmetric three dimensional way.

The operator connections: With the eight values of the bi-spinor field visualized on the cube's vertices we have a means to visualize the operators. The operators define linear connections between the field values. There are two distinct sets of connections.

- The 12 face diagonals of the cube $\longrightarrow$ *Pauli matrices*
- The 12 edges of the cube $\longrightarrow$ *Mass coupling term*

The Pauli matrices represent connections along the *face-diagonals* of the cube. This causes (1) that the cube is automatically separated into two (chiral) states which are each others parity inverse with respect to the center of the cube and (2) The phase of the fermion field is one-halve of the geometrical angle, as required for a spin $\frac{1}{2}$ field.

The connections along the *edges* of the cube represent the mass coupling between the two chiral states which is defined symmetrically in the x, y and z coordinates.

Omnidirectional rotation: This representation also solves the question of how the Abelian U(1) operator, which is effectively a two dimensional rotator, can work in a three dimensional space without having a preferred direction. We will see from the generators of rotation that it acts as an *omnidirectional* rotator which simultaneously rotates values on the faces of the cube in the outward (or inward) directions. Such an omnidirectional U(1) operator can only be defined by acting along the face-diagonals of the cube.

The 8×8 bi-spinor operator representation

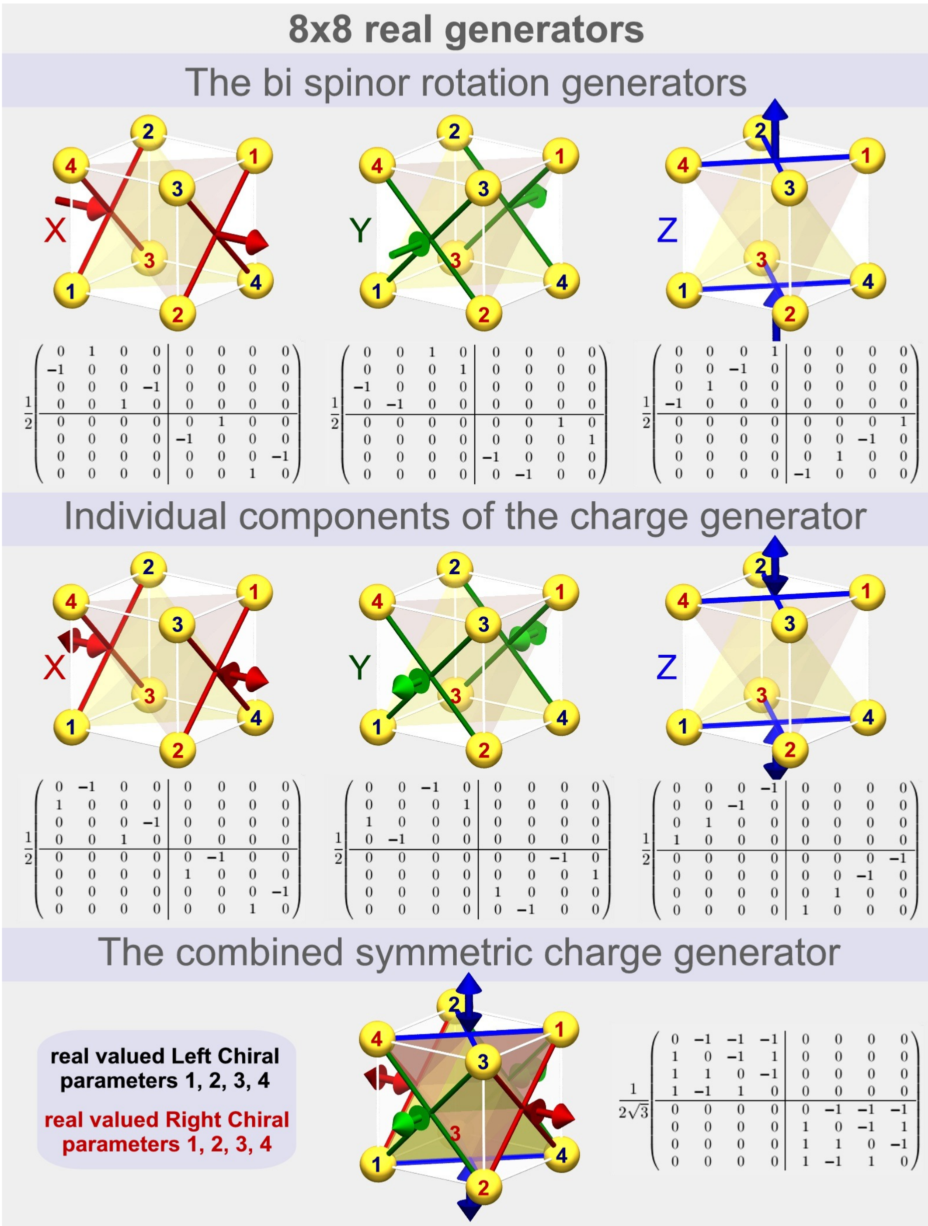

Figure 4: The generators of rotation and the generator of charge

The 8×8 bi-spinor operator representation

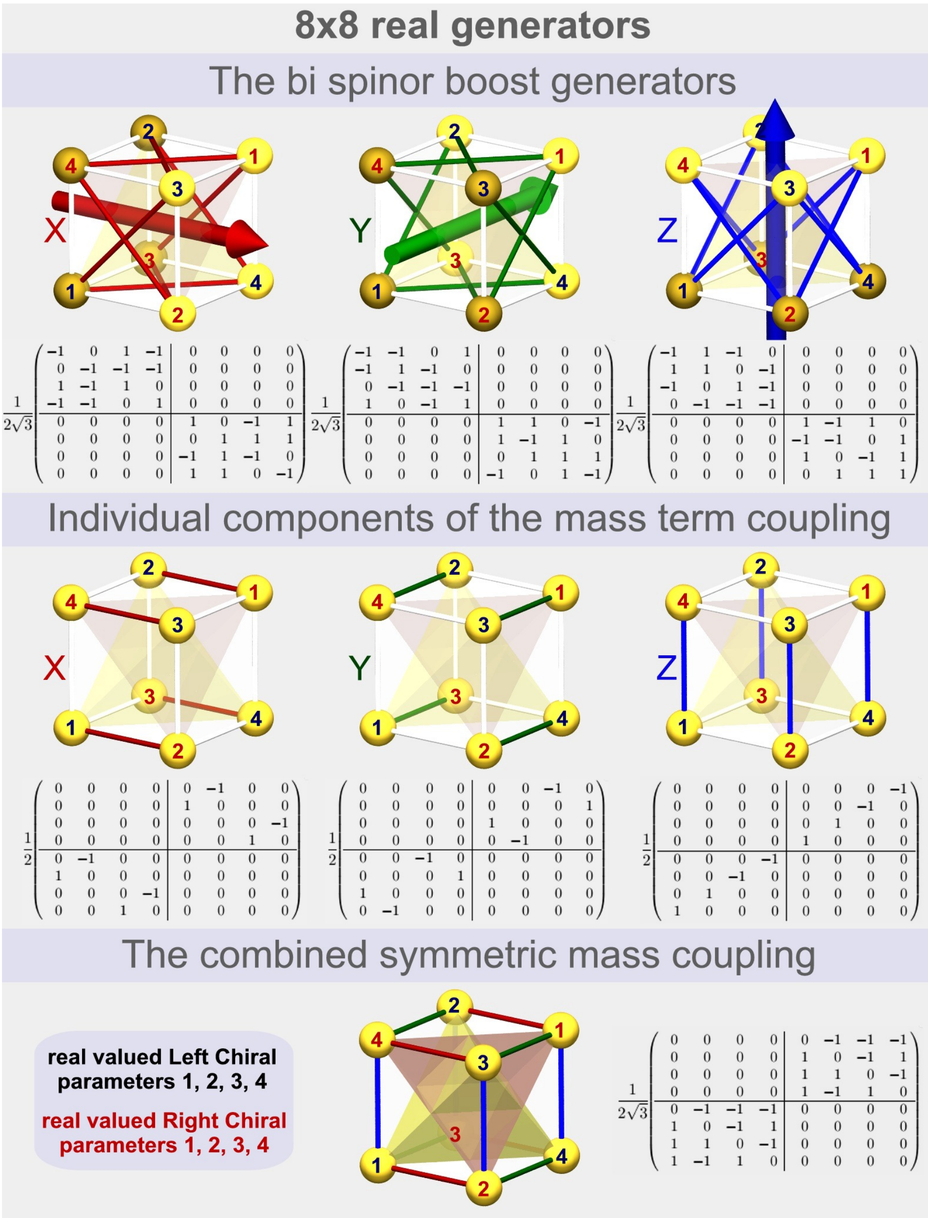

Figure 5: The generators of boosts and the mass coupling matrix

1.7 The 8×8 bi-spinor operator representation

Figures (4) and (5) show in detail the correspondence of the bi-spinor matrices at one hand and the lines connecting the parameters in the visualization.

Generally there's no reason for this level of detail since, at the Pauli matrix level, the representation is exactly the same as the standard representation. For instance the generators of rotation and boosts.

$$J^i \;=\; -\tfrac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad K^i \;=\; \tfrac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{21}$$

Or the matrices accompanying the electromagnetic interaction term and the mass term in the Dirac equation.

$$\begin{pmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix} eA^\mu \qquad \begin{pmatrix} 0 & \mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} m \tag{22}$$

These are the matrices shown in detail in figures (4) and (5). The lines between parameters correspond to pairs of coefficients which connect the parameters, either asymmetrically for the rotation generators and the matrices for eA and m, or symmetric for the boost generators.

$$\text{Asymmetric:} \quad \pm\begin{pmatrix} 0 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \qquad \text{Symmetric:} \quad \pm\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \tag{23}$$

The symmetric boost generators have also coefficients on the main diagonal, either +1 or -1. These are visualized by the color of the parameter. The color is lighter if the parameter will grow due to a boost in the corresponding x, y or z direction while the color is shown darker if the parameter will get smaller.

The matrices accompanying eA and m are symmetric in x, y and z due to the choice made for i. Both matrices are shown in their individual components and as complete matrices. The complete matrices are the normalized sums of the individual component matrices.

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2 The real symmetric Dirac equation

Chapter 2

Real Symmetric Representation

2.1 The real 4×4 spinor rotation generators

A good way to find the spatial symmetric, real valued 4×4 matrix equivalents of the Pauli matrices is to start with the complex generators of rotation. We'll find that their real valued representation is already symmetric in the x, y, z coordinate system. We obtain the real valued matrices by applying the standard substitution:

$$a + ib \quad = \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{24}$$

Substitution in the three complex spinor rotation generators given by $J_i{=}-\frac{i}{2}\sigma_i$ gives us the following 4×4 spinor rotation matrices:

The real valued 4×4 spinor rotation generators

$$-\tfrac{1}{2}\mathbf{i}\sigma_{x'}, \quad -\tfrac{1}{2}\mathbf{i}\sigma_{y'}, \quad -\tfrac{1}{2}\mathbf{i}\sigma_{z'} \quad = \tag{25}$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Where $\mathbf{i}$ is the 4×4 matrix representation of $i\sigma_o$ which is to be defined. Each of the real valued matrices does contain *two* generators of rotation.

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \exp \tfrac{\phi}{2} j = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \tag{26}$$

Using the three matrices we construct the general *spinor rotation operator*.

$$\exp(J^i\phi_i) \quad = \quad \exp \frac{1}{2}\begin{pmatrix} 0 & \phi_x & \phi_y & \phi_z \\ -\phi_x & 0 & -\phi_z & \phi_y \\ -\phi_y & \phi_z & 0 & -\phi_x \\ -\phi_z & -\phi_y & \phi_x & 0 \end{pmatrix} \tag{27}$$

This rotation operator is the 4×4 real valued matrix representation of the *quaternions*[1] first described (in their original form) by Hamilton in 1843.

[1]The quaternion coordinate system results in matrices which are easier to remember. A spatial coordinate system rotation $\{x', y', z'\} = \{z - y, x\}$ is used with regard to the standard Pauli matrices σ_i. We will omit the ' during the rest of the document.

2.2 Visualization of the rotation algebra

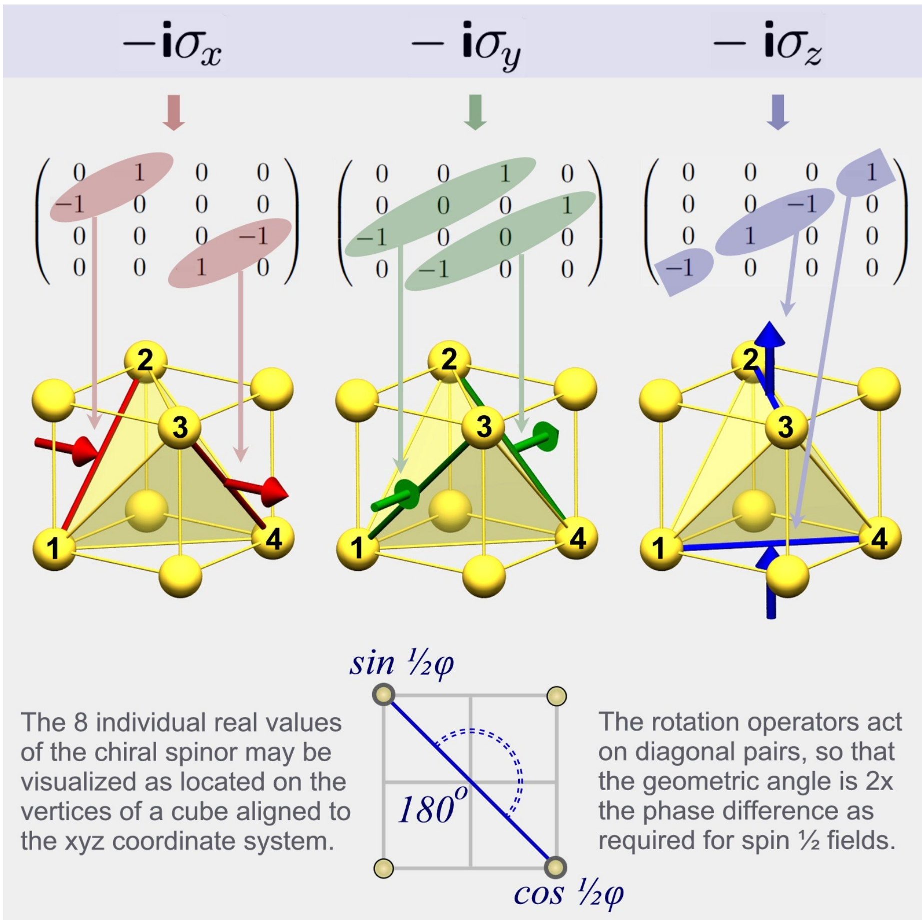

Figure 6: Rotation generator acting on the (left) chiral state.

Both the left and right chiral spinor become a 4×1 real valued array. The 8 values of these two arrays may be envisioned on the 8 vertices of a cube aligned with the coordinate system. The two chiral components are each other's *parity inverse* on the cube.

The generators of rotation pair *diagonally* located values at 180^o on the faces of the cube so that the spatial angle is *twice* the 90^o phase difference between the sine and the cosine, as is required for a spin $\frac{1}{2}$ representation.

2.3 The symmetric 3 dimensional generator of charge

A remarkable mathematical property of the U(1) operator is that it acts isotropically on spinors. It always rotates a spinor around the spinor's *own axis* completely independent of the direction in which the spinor points. It rotates *relative* with regard to the spinor instead of absolute around some fixed axis. For a spinor ξ_s pointing in the $\vec{s}$ direction we can therefor write.

$$e^{i\vec{s}\cdot\vec{\sigma}}\,\xi_s \;=\; e^{is}\,\xi_s \qquad\qquad s = \sqrt{s_x^2 + s_y^2 + s_z^2} \tag{28}$$

The reason for this is that we define the spinor ξ_s as being an eigenvector of the boost matrix $\vec{s}\cdot\vec{\sigma}$ in the $\vec{s}$ direction, and we therefor may replace the $\vec{s}\cdot\vec{\sigma}$ at the left hand side by the eigenvalue s at the right hand side.

$$(\vec{s}\cdot\vec{\sigma})\,\xi_s \;=\; s\,\xi_s \qquad\qquad s = \sqrt{s_x^2 + s_y^2 + s_z^2} \tag{29}$$

We see that we are allowed to associate a *geometric* meaning to the phase ϕ as the angle of rotation around the spinor axis by 2ϕ. The operator U(1) is therefor Abelian and apparently two dimensional since there is only one such axis, even in three dimensional space.

The goal is now to construct a matrix, symmetric in the x, y and z components, which acts as the generator of rotations of a spinor around it own axis. We start by using the *"complex-number-to-real-matrix"* rule (24) and apply it on the matrix $i\sigma_o$ which is the generator of the Abelian phase ϕ in the standard notation.

$$i\sigma^o \;=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{30}$$

This U(1) generator also contains two rotators as in the case of the spinor rotation generators. It is actually the same as J_x with the sign of one rotator reversed[2], as shown in figure 7. Both rotators now point outwards and in opposite directions.

[2] "*The same as J_z with one rotator reversed*" in Pauli's choice of coordinates

2.4 The symmetric charge generator algebra visualized

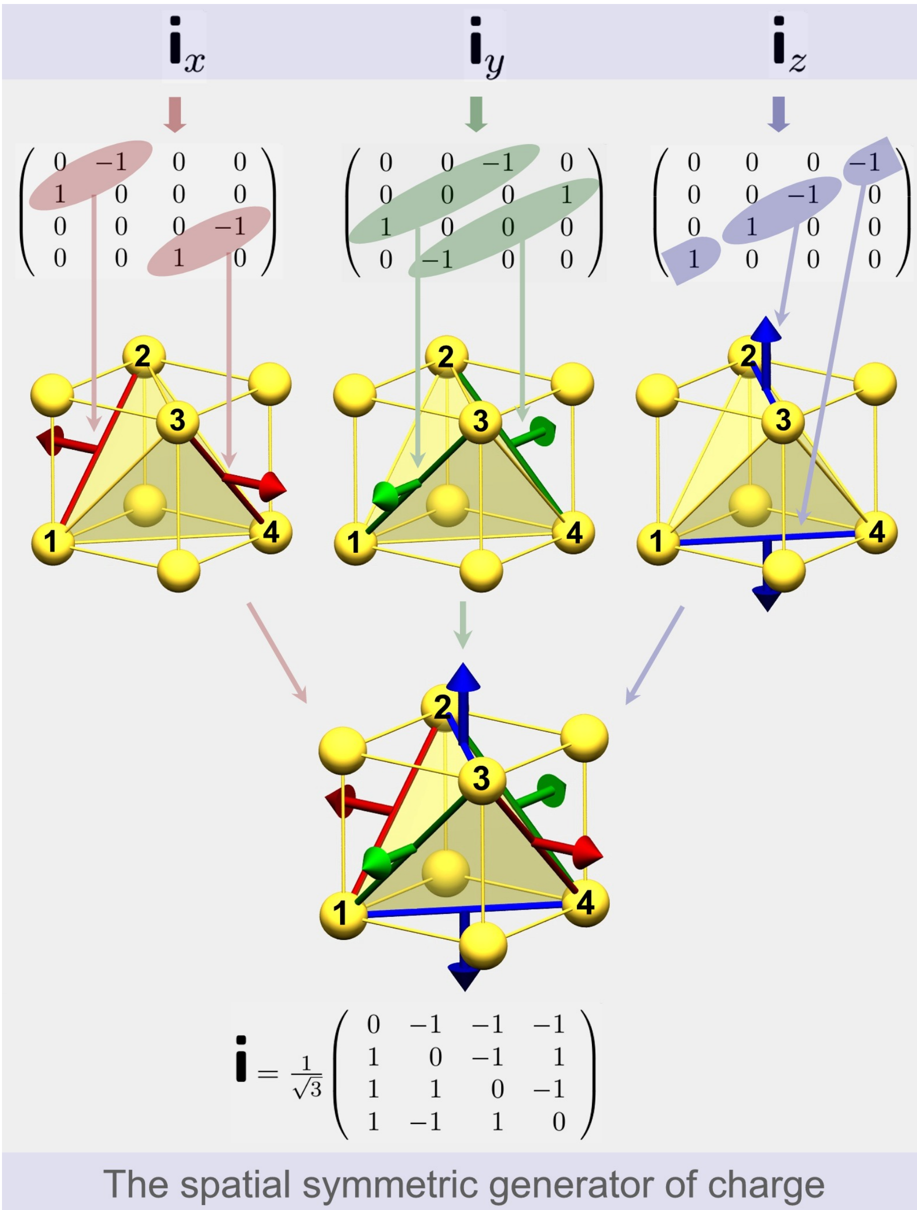

Figure 7: The charge generator as the averaged sum of $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$

This definition of U(1) is not symmetric in space and it leads to spin eigenstates which are not symmetric in x, y and z. We can obtain spin states which are direction independent by using the extra degrees in freedom given by the real 4×4 matrices. We can define one such "directional" $\mathbf{i}_i$ generator for each of the three coordinate axis. as shown in figure 7.

The two matrices $\mathbf{i}_y$ and $\mathbf{i}_z$ can not be represented[3] by 2×2 complex matrices. We can define a triplet of generators $J_i^{\rm rel}\,(=\frac{1}{2}\mathbf{i}_i)$. This triplet has the same commutation rules as the normal rotation generators $J_i^{\rm abs}\,(=-\frac{1}{2}\mathbf{i}\sigma_i)$.

$$\left[\, J_i^{\rm rel}\ ,\ J_j^{\rm rel} \right] \;=\; \varepsilon^{ijk}\, J_k^{\rm rel}, \qquad\qquad \left[\, J_i^{\rm abs}\ ,\ J_j^{\rm abs} \right] \;=\; \varepsilon^{ijk}\, J_k^{\rm abs} \tag{31}$$

Together however they *commute*. Relative rotation commutes with absolute rotation and therefor any linear combination of the generators of relative rotation $J_i^{\rm rel}$ commutes with any linear combination of the $J_i^{\rm abs}$

$$\left[\, J_i^{\rm rel}\ ,\ J_j^{\rm abs} \right] \;=\; 0, \qquad \text{for all } i \text{ and } j \tag{32}$$

We can use any such linear combination of the $\mathbf{i}_i$ as an Abelian charge operator and thus chose a symmetric one:

The 4×4 real valued, symmetric in space $U_{(1)}$ generator

$$\mathbf{i} \;=\; \frac{1}{\sqrt{3}}\,(\,\mathbf{i}_x+\mathbf{i}_y+\mathbf{i}_z\,) \;=\; \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \tag{33}$$

The symmetric one is the one which will give us symmetric spin eigen states. For all matrices $\mathbf{i}^2 = \mathbf{i}_x^2 = \mathbf{i}_y^2 = \mathbf{i}_z^2 = -\mathbf{1}$ is true. This U(1) operator has no preferred direction in the representation in the sense that it has components in the representation rotating in all directions equally. See both figures (7) and (8). However, it rotates the spinor eigenstates, which *do* have a specific direction, around their own axis. Therefor it is an Abelian, "two dimensional" operator in a three dimensional space.

[3] These matrices can be represented by using the complex conjugate operator * which selectively inverts the sign of two of the four spinor parameters, the imaginary ones. We find that σ^2* and $i\sigma^2*$ correspond with the matrices $\mathbf{i}_x$ and $\mathbf{i}_y$. These two operators do not commute with i. The three commute instead as a triplet of *relative* rotation operators. The generators σ^2* and $i\sigma^2*$ always rotate a spinor *perpendicular* relative to the spinor's axis.

2.5 The charge generator acting on the total bi-spinor

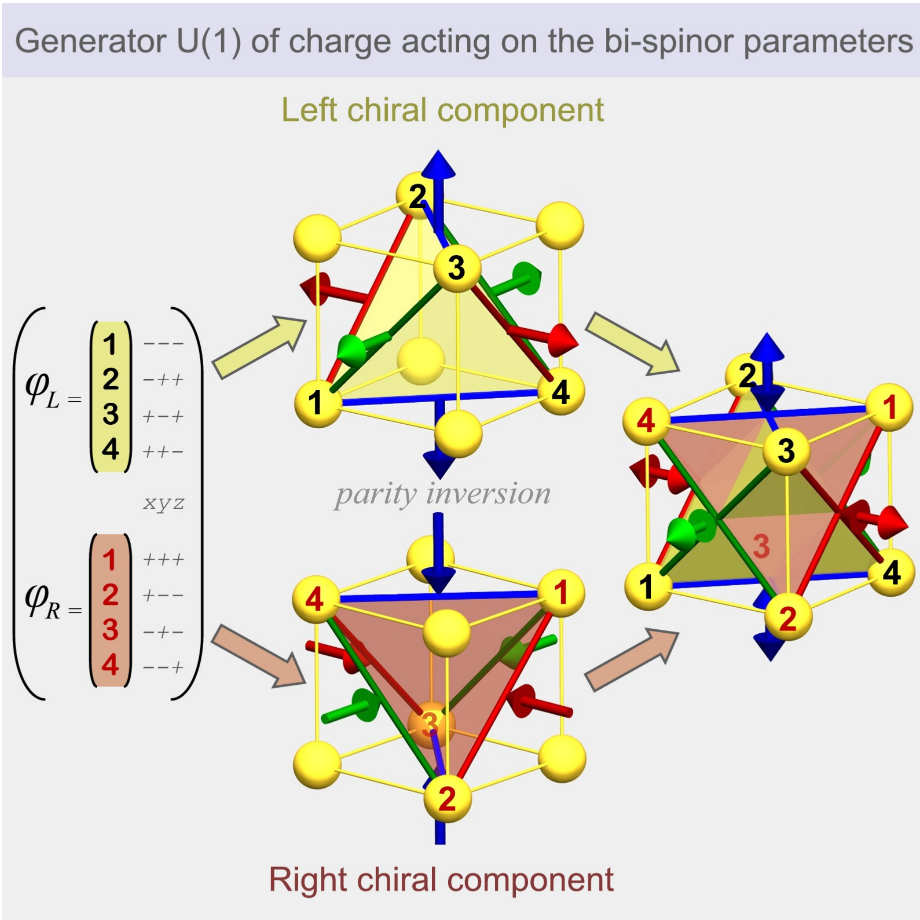

Figure 8: The real chiral spinor shown under U(1) time evolution.

Figure (8) shows the locations of the individual chiral bi-spinor parameters. The arrows show how the electromagnetic U(1) time evolution operator acts on the chiral bi-spinor. All the individual sine/cosine interactions are directed *outwards* in case of the left chiral component of an electron and *inwards* for the right chiral one. For a plane wave at rest in a static field:

$$\text{interaction}\left(\frac{\partial\psi}{\partial t}\right) \;=\; U_{(1)}\psi \;=\; -\,\mathbf{i}eA_o\psi \qquad (34)$$

2.6 The real valued symmetric Dirac equation

With the definition of $\mathbf{i}$and the three $\mathbf{i}\sigma_i$ from the rotation generators we have already made all the definitions required to define the fully symmetric, real valued replacement of the Dirac equation. We can replace the standard complex, asymmetric Dirac equation

$$i\gamma^\mu (\partial_\mu + ieA_\mu) \psi = m\psi \tag{35}$$

with the real valued, spatial symmetric form of the Dirac equation with a virtual identical notation.

The real valued, spatial symmetric Dirac equation

$$\boxed{\mathbf{i}\gamma^\mu (\partial_\mu + \mathbf{i}eA_\mu) \psi = m\psi} \tag{36}$$

All we needed to do here was just replacing both i with the real matrix $\mathbf{i}$. When we write out the gamma matrices into explicit Pauli matrices the real valued Dirac equation

$$\begin{pmatrix} 0 & \mathbf{i}\sigma^\mu \\ \mathbf{i}\tilde{\sigma}^\mu & 0 \end{pmatrix} \left(\partial_\mu + \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} e\mathrm{A}_\mu \right) \psi = m\psi \tag{37}$$

All the new definitions we need to use are here given by the four matrices.

$$\mathbf{i}\sigma_o,\ \mathbf{i}\sigma_x,\ \mathbf{i}\sigma_y,\ \mathbf{i}\sigma_z \ = \tag{38}$$

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We can simply use $\mathbf{i}$ for the first matrix since σ_o remains the unity matrix.

2.7 Visualization of the real symmetric Dirac equation

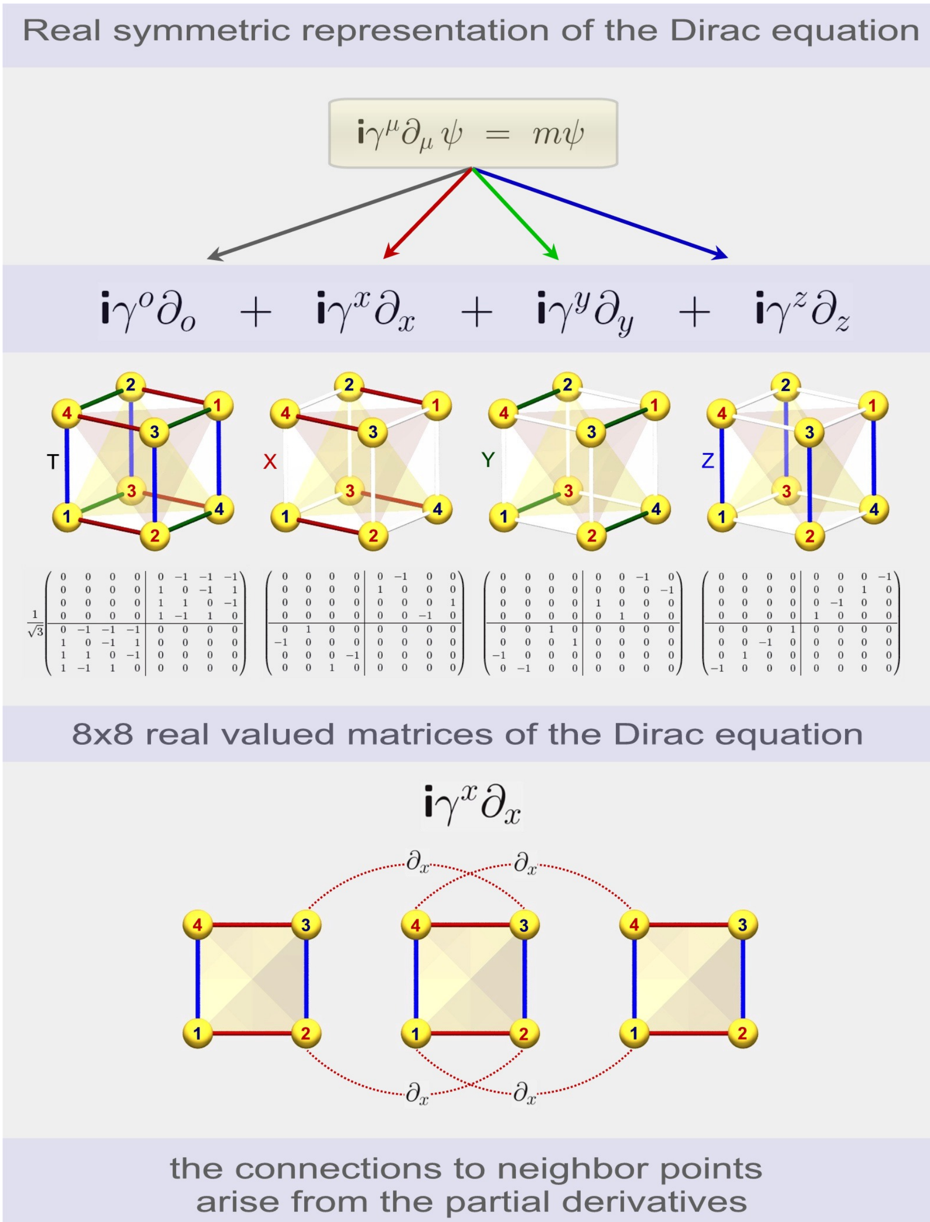

Figure 9: The real symmetric Dirac equation and its derivative matrices.

2.8 The geometrical meaning of the matrices

The real symmetric representation allows us to understand the geometrical meaning of the matrices used in the equation, as shown in figure (9).

First of all we see that the spatial connections, visualized by the red, green and blue lines, point in the direction associated with the indices of $\mathbf{i}\gamma^i\partial_i$ and thus in the same direction as the derivatives are taken over the field.

We want to understand how the spatial derivative matrices combine with the time derivative matrix in order to produce the equation of motion. We do so by starting first with simpler equations.

2d linearized Klein Gordon equation

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\right]\psi \;=\; m\psi \tag{39}$$

2d real valued Dirac equation

$$\left[\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\frac{\partial}{\partial x}\right]\psi \;=\; m\psi \tag{40}$$

4d real valued Dirac equation

$$\left[\mathbf{i}\gamma^o\frac{\partial}{\partial t} + \mathbf{i}\gamma^x\frac{\partial}{\partial x} + \mathbf{i}\gamma^y\frac{\partial}{\partial y} + \mathbf{i}\gamma^z\frac{\partial}{\partial z}\right]\psi \;=\; m\psi \tag{41}$$

In all three cases above the square of expression between brackets $-[\cdots]^2$ becomes the d'Alembertian so that all individual field components obey the Klein Gordon equation.

$$-\Big[\cdots\Big]^2 = \partial_\mu\partial^\mu \qquad \longrightarrow \qquad \left(\partial_\mu\partial^\mu + m^2\right)\psi = 0 \tag{42}$$

The matrices are anti commuting. The time derivative matrix must be unitary and antisymmetric in such a way that $(\mathbf{i}\gamma^o)^2 = -1$ while the spatial derivative matrices must be symmetric instead so that $(\mathbf{i}\gamma^i)^2 = 1$

2.9 Left and right moving sub-equations of motion

The two simplest possible equations of motion are given below. The field ψ_L can be any arbitrary field which shifts to the left with velocity c and ψ_R is an equivalent right shifting field.

$$\frac{\partial \psi_L}{\partial t} - c\frac{\partial \psi_L}{\partial x} = 0 \qquad\qquad -\frac{\partial \psi_R}{\partial t} - c\frac{\partial \psi_R}{\partial x} = 0 \tag{43}$$

One way to accommodate both the left and right moving solutions is the 2d second order classical wave equation. This equation owes its bidirectional nature to the parameter c^2 where c can be both positive and negative.

$$\frac{\partial^2 \psi}{\partial t^2} - c^2\frac{\partial^2 \psi}{\partial x^2} \tag{44}$$

But there is another possibility which combines both the linear nature of (43) and the classical wave equation given above. We do so by retaining the left and right moving waves as two separated components of a two component wave function.

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} + c \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \tag{45}$$

This expression combines both equations (43) into a single equation. If we apply the operator between square brackets twice (by squaring it) then we recover the classical wave equation (44) and so both the left and right going components ψ_L and ψ_R obey the classical wave equation.

We obtain the 2d linearized Klein Gordon equation (39) if we couple the two independent equations with a mass term $m\psi$. From the classical wave equation and its Greens function we know that the term on the right hand side is actually the *source* of the wave function ψ. In the case of the Klein Gordon equation the field becomes its own source, the right moving channel ψ_R becomes a source for the left moving channel ψ_L and visa versa. The resulting field can propagate at any speed between $+c$ and $-c$. A typical plane wave solution for this equation has the form of a sine/cosine pair, the real valued equivalent of $\exp(-iEt + ipx)$.

The following equation, the 2d real valued Dirac equation (40) has four parameters, two involved in left shifting and two involved in right shifting.

Left and right moving sub-equations of motion

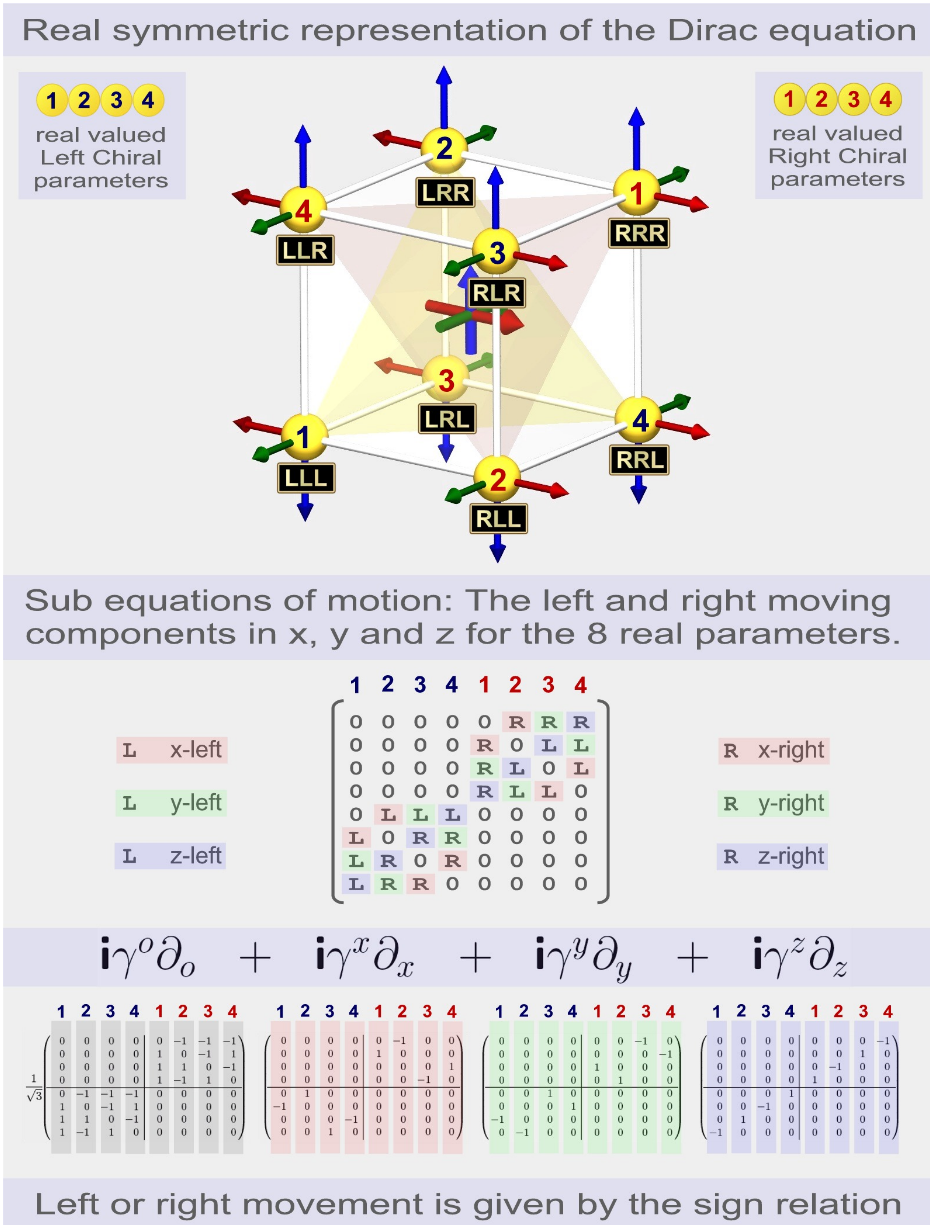

Figure 10: The split into left shifting and right shifting sub-equations.

We can now see how the full, 4d real valued Dirac equation is composed out of left shifting and right shifting sub-equations in the three spatial directions x, y and z.

Figure (10) shows how each of the eight parameters is involved in three shift sub-equations, one each in the x, y and z direction indicated by the red, green and blue arrows. The parameters are labeled with the corresponding shifting directions, left or right, per coordinate axis.

The eight vector currents $\psi_e\gamma^\mu\psi_e$, corresponding to the eight parameters in isolation, point from the center of the cube outwards to the eight vertices associated with the individual parameters. We see these current directions back in the shift sub-equations in which the parameters are involved.

The 8×8 matrix in the middle shows where the shift sub-equations are located. The x, y or z directions correspond with the non-zero elements of the x, y and z matrices. The shift direction, left or right, is determined by the sign relation of the time derivative matrix and the corresponding space derivative matrix. Equal signs correspond to right shifting while unequal signs correspond with left shifting.

2.10 Gradient and divergence sub-equations of motion

The linearization separates the d'Alembertian $\partial_\mu\partial^\mu$ in a gradient and divergence part. Figure (11) highlights how this works for one of the eight parameters: The Left chiral parameter 1 which is labeled LLL.

The x, y and z components of the gradient of this parameter go to its three nearest neighbors in the x, y and z direction. The place where this happens is in the first column of the matrices. The following application of the matrix/differential operator then combines the derivatives of these three values back again via the divergence operation. The divergence takes place in the first row of the matrices.

The columns are always associated with the gradient operations while the rows are associated with the divergence operatations. The intermediate gradient results get mixed with other gradient results but the non-commutative nature of the matrices separates them again by eliminating the cross-terms.

The gradient time derivative is, due to the symmetric shifting sub-equations, distributed along equally with the three spatial derivatives.

Gradient and divergence sub-equations of motion

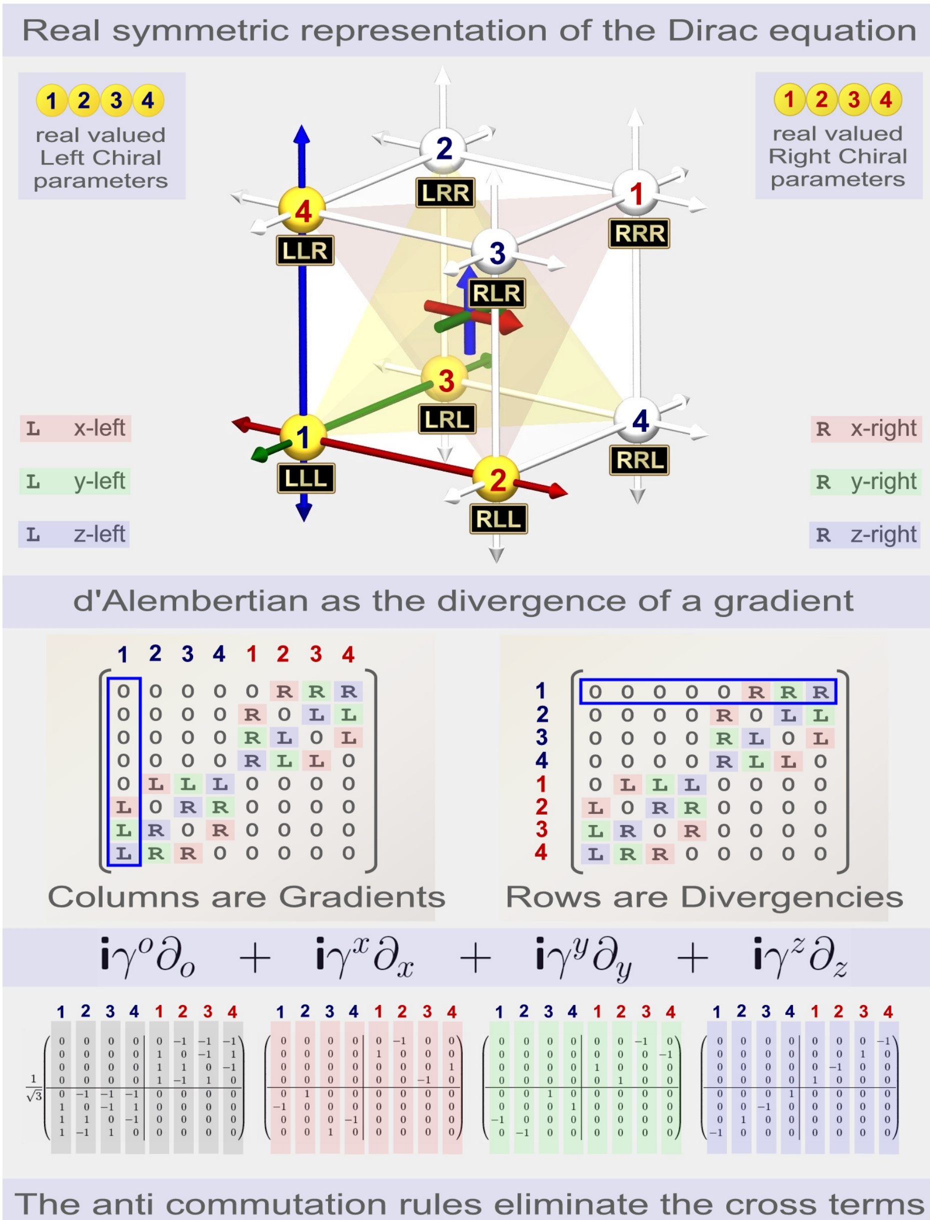

Figure 11: The split of the d'Alembertian in gradients and divergencies.

2.11 The real valued Pauli and Gamma matrices

To complete the definition of the matrices we obtain the 4×4 real valued replacements of the Pauli matrices from the now know $\mathbf{i}\sigma_\mu$ matrices by simply multiplying these with -**i**. We obtain,

The spatial symmetric, real valued Pauli matrices

$$\sigma_x,\ \sigma_y,\ \sigma_z \ \ = \tag{46}$$

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix},\ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix},\ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The matrix σ_o remains the unity matrix. All Pauli matrices σ_μ are symmetric while all matrices $\mathbf{i}\sigma_\mu$ are anti-symmetric so that the standard expressions in the complex representation

$$\sigma_\mu^\dagger = \sigma_\mu, \qquad (i\,\sigma_\mu)^\dagger = -i\,\sigma_\mu \tag{47}$$

become:

$$\sigma_\mu^\mathsf{T} = \sigma_\mu, \qquad (\mathbf{i}\,\sigma_\mu)^\mathsf{T} = -\mathbf{i}\,\sigma_\mu \tag{48}$$

in the real valued representation. The main diagonal elements of all Pauli matrices are non-zero. This is what causes the base vectors $\bar{\psi}_e\gamma^i\psi_e$ to be non-degenerate and spatially symmetric as we shall see in the next section. The trace of each of the three Pauli matrices σ_x, σ_y and σ_z is zero.

At the gamma matrix level nothing changes despite the rather different appearance of the Pauli matrices themself.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{49}$$

The standard definition of the γ^5 matrix leads to the familiar chiral result.

$$\gamma^5 = \mathbf{i}\,\gamma^o\gamma^x\gamma^y\gamma^z = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{50}$$

This completes the set of gamma matrices in the chiral representation.

2.12 The Lie algebra of the Lorentz group

The standard definitions[4] also apply for the Lie algebra of the Lorentz group. We'll list the basic equations here. The anti commutator of the gamma matrices produces the metric.

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{51}$$

While the commutator produces the representation of the Lorentz algebra. Note that we'll use the convention that not uses the number i since the goal is to treat the representation as an entirely real valued one.

$$J^{\mu\nu} \;=\; \tfrac{1}{4}\,[\,\gamma^\mu\,,\,\gamma^\nu\,] \tag{52}$$

This means that all generators generators of rotations and boosts at the chiral spinor level.

$$J_i = -\tfrac{1}{2}\begin{pmatrix} \mathbf{i}\sigma_i & 0 \\ 0 & \mathbf{i}\sigma_i \end{pmatrix} \qquad\qquad K_i = -\tfrac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{53}$$

The generators obey the Lie algebra of the Lorentz group.

$$[\,J_i, J_j\,] = \varepsilon^{ijk}\,J_k \qquad [K_i, K_j] = -\varepsilon^{ijk}\,J_k \qquad [\,J_i, K_j\,] = \varepsilon^{ijk}\,K_k \tag{54}$$

where ε^{ijk} is the totally anti-symmetric symbol. For the representation algebra we can simply replace the i with the real matrix $\mathbf{i}$.

$$J_i^+ \;=\; J_i + \mathbf{i}K_i \qquad\qquad J_i^- \;=\; J_i - \mathbf{i}K_i \tag{55}$$

The commutation rules stay the same.

$$\left[\,J_i^+, J_j^+\,\right] = \varepsilon^{ijk}\,J_k^+ \qquad \left[\,J_i^-, J_j^-\,\right] = \varepsilon^{ijk}\,J_k^- \qquad \left[\,J_i^+, J_j^-\,\right] = 0 \tag{56}$$

[4] note that the definitions of both the generators and commutators are all in the real valued form as in classical relativistic dynamics.

2.13 The 3d symmetric mass generator algebra visualized

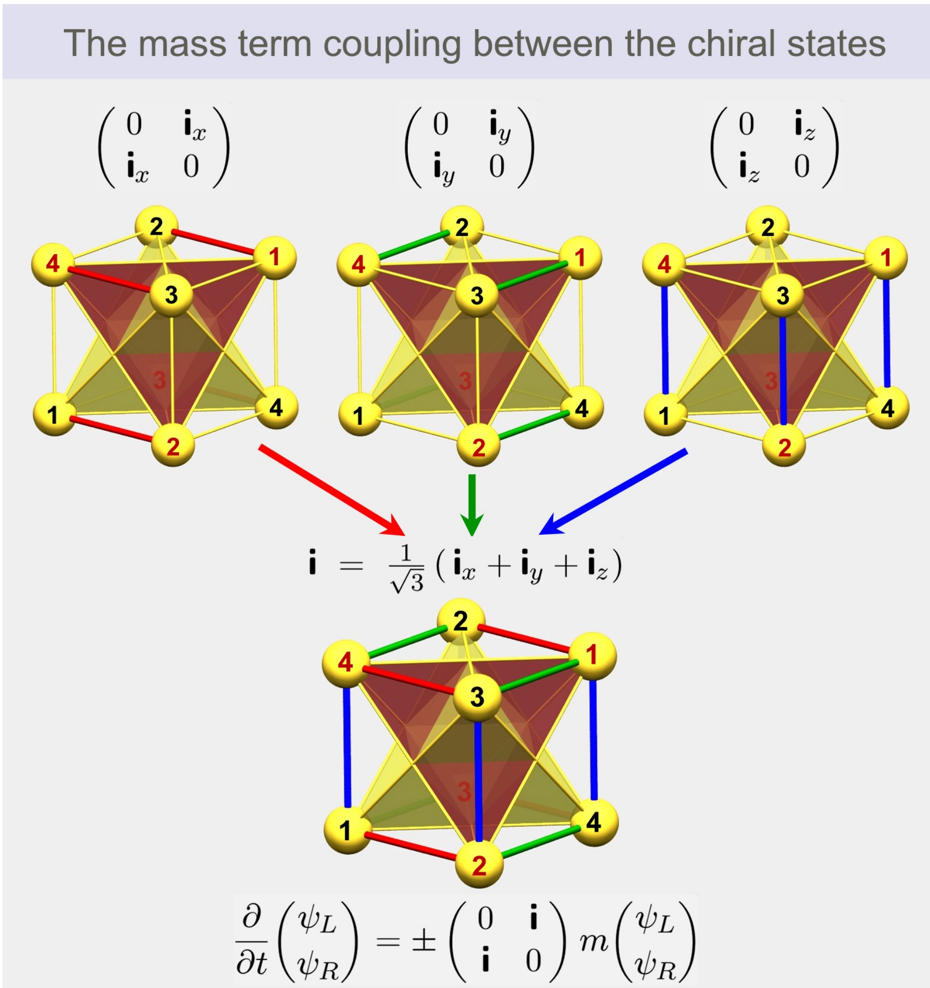

Figure 12: The submatrices couple the chiral states along their own axis.

For a free field in its restframe, which has no spatial derivatives and no interaction, we can rewrite the Dirac equation as.

$$H = \frac{\partial}{\partial t} = - \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} m \tag{57}$$

In this form it represents the Hamiltonian time evolution generator.

The Hamiltonian operator to determine ψ for any time t becomes.

$$H(t) \;=\; \exp\left\{-\begin{pmatrix}0 & \mathbf{i}\\ \mathbf{i} & 0\end{pmatrix} mt\right\} \;=\; \begin{pmatrix}\mathbf{1} & 0\\ 0 & \mathbf{1}\end{pmatrix}\cos(mt) - \begin{pmatrix}0 & \mathbf{i}\\ v & 0\end{pmatrix}\sin(mt) \tag{58}$$

The matrix $\mathbf{i}$connects the two chiral states via the mass parameter m. The chiral states are each others parity inverse. The submatrices $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ couple the chiral states over the x, y and z-axis respectively. This can be checked by looking at figure (7) and moving one of the two interacting vertices via the center of the cube to the location at the other side.

2.14 The base-vectors of the bispinor field

The chiral bispinor contains a set of 8 real parameters. These parameters can be visualized as being located on the eight vertices of a cube, with the locations determined by the vector currents.

$$\text{cube vertex location} \;=\; \bar{\psi}_e\,\gamma^i\,\psi_e \tag{59}$$

Each of the ψ_e has only one non-zero parameter. This means that we only have to look at the main diagonals of the Pauli matrices since the non-diagonal coefficients are multiplied by zero.

$$\begin{aligned}\mathrm{diag}(\sigma_x) &= (\ 1,\ \ 1,-1,-1)\\ \mathrm{diag}(\sigma_y) &= (\ 1,-1,\ \ 1,-1)\\ \mathrm{diag}(\sigma_z) &= (\ 1,-1,-1,\ \ 1)\end{aligned} \tag{60}$$

These diagonal coefficients[5] determine the location of the parameters:

$$\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}\psi_{L1}\\ \psi_{L2}\\ \psi_{L3}\\ \psi_{L4}\\ \\ \psi_{R1}\\ \psi_{R2}\\ \psi_{R3}\\ \psi_{R4}\end{pmatrix} \qquad \text{visualized at:} \qquad \begin{array}{lll} x=-1 & y=-1 & z=-1\\ x=-1 & y=\ \ 1 & z=\ \ 1\\ x=\ \ 1 & y=-1 & z=\ \ 1\\ x=\ \ 1 & y=\ \ 1 & z=-1\\ \\ x=\ \ 1 & y=\ \ 1 & z=\ \ 1\\ x=\ \ 1 & y=-1 & z=-1\\ x=-1 & y=\ \ 1 & z=-1\\ x=-1 & y=-1 & z=\ \ 1\end{array} \tag{61}$$

[5] All diagonals from the four σ^μ combined give the 4×4 Hadamart matrix.

2.15 The spinor eigenstates under U(1)

We will discuss here how to derive the spin eigen-states. We will do so for the most general case of a normalized arbitrary U(1) operator using:

$$\mathbf{i} \;=\; \hat{c}_x\,\mathbf{i}_x + \hat{c}_y\,\mathbf{i}_y + \hat{c}_z\,\mathbf{i}_z \tag{62}$$

We'll then find our symmetric spin states by setting all $\hat{c}$ to $\sqrt{1/3}$ but we can also retrieve the asymmetric eigen-states of the complex equation where only one of the three $\hat{c}$ is non-zero.

The spin states are the eigenvectors of the Pauli matrices. The ones with eigenvalue +1 are the spin-up states while the spin-down states have -1 as eigenvalue. The Pauli matrices thus multiply the up-states by 1 and the down-states by -1.

This is why we can use the Pauli matrices to extract the vector current $\bar{\psi}\gamma^\mu\psi$ and axial current $\bar{\psi}\gamma^\mu\gamma^5\psi$. It is also the reason that we can use the Pauli matrices as the generators of boosts since the boost of a spin in a direction (anti-)parallel to its spin-direction only changes the magnitude of the spin and not its direction.

For the light-like transforming chiral components a boost comes down to a relativistic doppler shift representing the transformed phase change rate due to the boost. The operators $\exp(\pm\frac{1}{2}\vartheta^i\sigma^i)$ are the expressions for the relativistic doppler shift due to a rapidity change $\pm\vartheta^i$ with an extra factor $\frac{1}{2}$ in the argument because the fermions transform as spin $\frac{1}{2}$ particles.

The complex Pauli matrices each have two spin states, one spin-up state (eigenvalue +1) and one down (eigenvalue -1). The real valued 4 by 4 Pauli matrices have four eigenvectors: two degenerate orthogonal eigenvectors for each spin state, up and down so that we can write.

$$\xi(t) = \xi_1\cos(\omega t) - \xi_2\sin(\omega t) \qquad \text{versus} \qquad \xi(t) = \xi\, e^{-i\omega t} \tag{63}$$

The spinning $\xi(t)$ remains an eigenvector for any time t. This is why we need 2 orthogonal eigenvectors to allow the real valued spin to spin.

2.16 The fully symmetric electron spin eigenstates

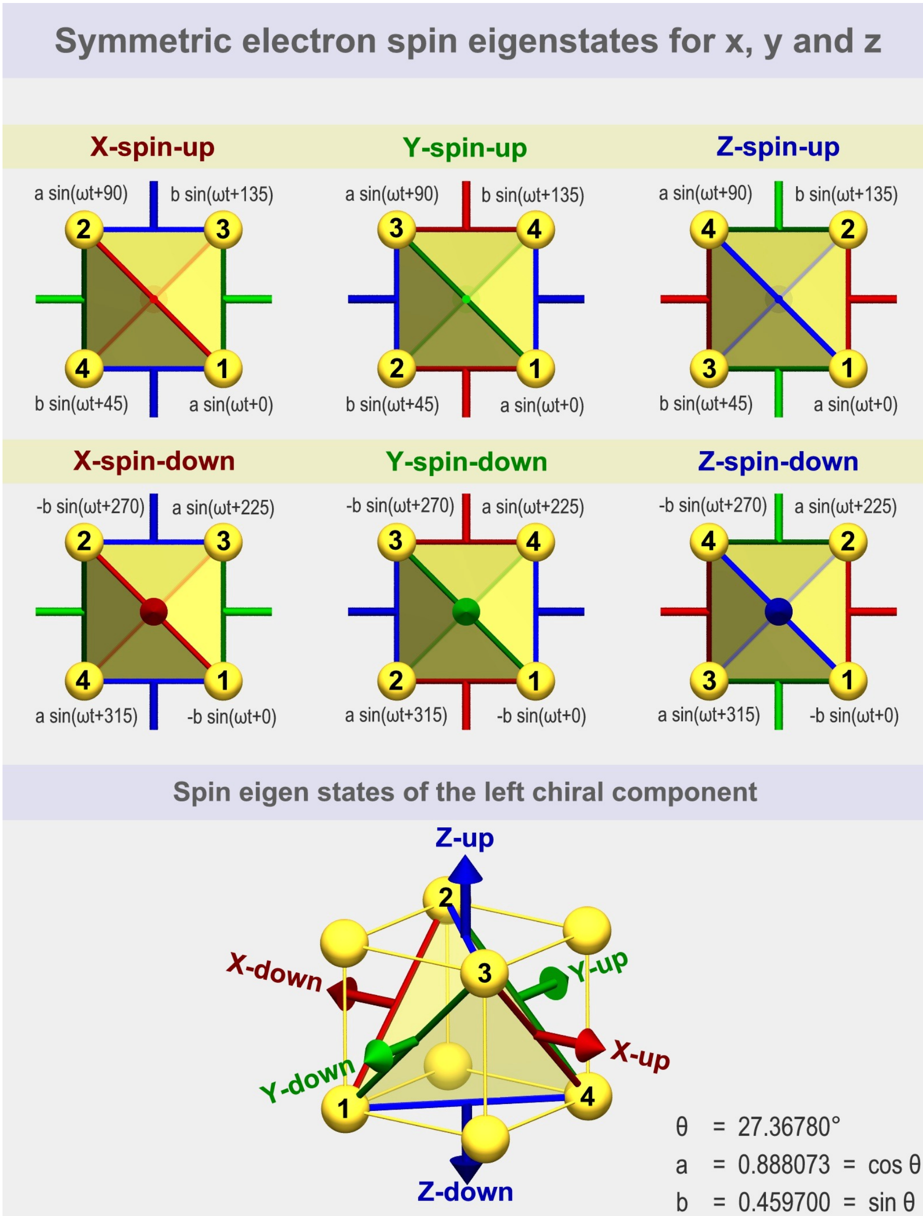

Figure 13: The symmetric spin eigenstates (left chiral component)

Now, working out the spin states for the three Pauli matrices in the general case gives us for the left chiral component in the rest frame:

The up spinors

$$\left\{ \begin{pmatrix} a_x \sin\frac{1}{2}(\omega t - \phi_x) \\ a_x \cos\frac{1}{2}(\omega t - \phi_x) \\ b_x \cos\frac{1}{2}(\omega t + \phi_x) \\ b_x \sin\frac{1}{2}(\omega t + \phi_x) \end{pmatrix}, \begin{pmatrix} a_y \sin\frac{1}{2}(\omega t - \phi_y) \\ b_y \sin\frac{1}{2}(\omega t + \phi_y) \\ a_y \cos\frac{1}{2}(\omega t - \phi_y) \\ b_y \cos\frac{1}{2}(\omega t + \phi_y) \end{pmatrix}, \begin{pmatrix} a_z \sin\frac{1}{2}(\omega t - \phi_z) \\ b_z \cos\frac{1}{2}(\omega t + \phi_z) \\ b_z \sin\frac{1}{2}(\omega t + \phi_z) \\ a_z \cos\frac{1}{2}(\omega t - \phi_z) \end{pmatrix} \right\} \tag{64}$$

The down spinors

$$\left\{ \begin{pmatrix} -b_x \cos\frac{1}{2}(\omega t + \phi_x) \\ -b_x \sin\frac{1}{2}(\omega t + \phi_x) \\ a_x \sin\frac{1}{2}(\omega t - \phi_x) \\ a_x \cos\frac{1}{2}(\omega t - \phi_x) \end{pmatrix}, \begin{pmatrix} -b_y \cos\frac{1}{2}(\omega t + \phi_y) \\ a_y \cos\frac{1}{2}(\omega t - \phi_y) \\ -b_y \sin\frac{1}{2}(\omega t + \phi_y) \\ a_y \sin\frac{1}{2}(\omega t - \phi_y) \end{pmatrix}, \begin{pmatrix} -b_z \cos\frac{1}{2}(\omega t + \phi_z) \\ a_z \sin\frac{1}{2}(\omega t - \phi_z) \\ a_z \cos\frac{1}{2}(\omega t - \phi_z) \\ -b_z \sin\frac{1}{2}(\omega t + \phi_z) \end{pmatrix} \right\} \tag{65}$$

The values of the a_i, b_i and ϕ_i parameters follows from the three $\hat{c}_i$ as.

$$\begin{aligned} a_x &= \sqrt{\tfrac{1}{2}(1+c_x)}, & b_x &= \sqrt{\tfrac{1}{2}(1-c_x)}, & \phi_x &= \arctan\left(\frac{c_y}{c_z}\right) \\ a_y &= \sqrt{\tfrac{1}{2}(1+c_y)}, & b_y &= \sqrt{\tfrac{1}{2}(1-c_y)}, & \phi_y &= \arctan\left(\frac{c_z}{c_x}\right) \\ a_z &= \sqrt{\tfrac{1}{2}(1+c_z)}, & b_z &= \sqrt{\tfrac{1}{2}(1-c_z)}, & \phi_z &= \arctan\left(\frac{c_x}{c_y}\right) \end{aligned} \tag{66}$$

The rotators are pointing in the required direction and they posses the factor $\frac{1}{2}$ between the *geometrical angle* of rotation and the *phase* of the wave function as required for spin $\frac{1}{2}$ particles.

The chiral component are equal in the rest frame ($\psi_L = \psi_R$) in case of the electron and ($\psi_L = -\psi_R$) in case of the positron. This factor -1 which amounts to a 180^o phase shift and a 360^o geometrical angle makes the difference between a negative charge and a positive charge.

Figure 13 shows the relative angles for the left chiral spinor as 0^o, 45^o, 90^o and 135^o increasing clockwise for the spin up state of the electron, just as we would like for a spin one-half particle. The images are all seen from the negative side of an axis in the direction of the positive side.

The electron's spin down states rotate the other way around with an additional feature: Two of the values on the cube are multiplied by -1 representing a 360^o geometrical angle. This feature is specific for all spin-down states for both the electron and positron and is related to the negative eigenvalue of the states. The positron spin states rotate in the opposite direction as those of the electron states, as already shown in figure 3.

The values of the three $\hat{c}$ are $\sqrt{1/3}$ in our fully symmetric case where all components are the same. This leads to the following numerical values.

$$\begin{aligned} a &= 0.8880738339 &= \cos\theta \\ b &= 0.4597008433 &= \sin\theta \end{aligned} \tag{67}$$

The major components (a) and the minor components (b) rotate both in the same direction, the direction of the spin, even though the operator tries to rotate the minor component in the other direction. The coupling with the larger major component forces the minor component to rotate in the same direction.

$$\text{U(1) operator: } \Leftarrow\!\!\!\Updownarrow\!\!\!\Rightarrow \qquad \text{spin state: } \begin{pmatrix} \Uparrow \\ \uparrow \end{pmatrix} \tag{68}$$

The spin states of the original complex Dirac equation are obtained from the general expression with $\hat{c}_x, \hat{c}_y, \hat{c}_z = 0, 0, 1$. We obtain for the z-state:

$$\begin{aligned} a &= 1.0000000000 \\ b &= 0.0000000000 \end{aligned} \tag{69}$$

2.17 The complex dot products and real spinors

Complex dot products such as $\psi_L^\dagger \psi_L$ are simplified to the functionally equivalent real valued dot products $\psi_L^{\mathsf{T}} \psi_L$. Where † denotes the Hermitian transpose and T denotes the normal real transpose. This is directly obvious in the case that both terms are identical since.

$$\begin{array}{lcl} \psi_L^{\mathsf{T}}\psi_L & = & \psi_{L1}^2 + \psi_{L2}^2 + \psi_{L3}^2 + \psi_{L4}^2 \\ \psi_R^{\mathsf{T}}\psi_R & = & \psi_{R1}^2 + \psi_{R2}^2 + \psi_{R3}^2 + \psi_{R4}^2 \end{array} \tag{70}$$

For instance the invariance of $\psi_L^\dagger \psi_L$ under a phase shift $\exp i\phi$ as given by,

$$\Big(e^{i\phi}\ \psi\Big)^\dagger \Big(e^{i\phi}\ \psi\Big) \quad = \quad \psi^\dagger\ e^{-i\phi}\ e^{i\phi}\ \psi \quad = \quad \psi^\dagger \psi \tag{71}$$

becomes in the real valued matrix case the invariance of $\psi_L^{\mathsf{T}} \psi_L$ under the (Abelian) phase shift $\exp \mathbf{i}\phi$

$$\Big(e^{\mathbf{i}\phi}\ \psi\Big)^{\mathsf{T}} \Big(e^{\mathbf{i}\phi}\ \psi\Big) \quad = \quad \psi^{\mathsf{T}}\ e^{-\mathbf{i}\phi}\ e^{\mathbf{i}\phi}\ \psi \quad = \quad \psi^{\mathsf{T}} \psi \tag{72}$$

This because $\mathbf{i}$ is an *anti-symmetric* matrix and thus.

$$(\mathbf{i}\psi)^{\mathsf{T}} \quad = \quad \psi^{\mathsf{T}}\mathbf{i}^{\mathsf{T}} \quad = \quad -\,\psi^{\mathsf{T}}\mathbf{i} \tag{73}$$

Since the Pauli matrices σ^μ are all *symmetric* matrices and the $\mathbf{i}\sigma^\mu$ are all *anti symmetric* matrices we can straightforwardly reuse the following left versus right side multiplication rules.

$$\begin{array}{lclcl} (\ \sigma^\mu\psi)^\dagger & = & +\,\psi^\dagger(\ \sigma^\mu) & \text{becomes:} & (\ \sigma^\mu\psi)^{\mathsf{T}} \ = \ +\,\psi^{\mathsf{T}}(\ \sigma^\mu) \\ (i\sigma^\mu\psi)^\dagger & = & -\,\psi^\dagger(i\sigma^\mu) & \text{becomes:} & (\mathbf{i}\sigma^\mu\psi)^{\mathsf{T}} \ = \ -\,\psi^{\mathsf{T}}(\mathbf{i}\sigma^\mu) \end{array} \tag{74}$$

The (anti-) symmetry determines the sign. To appreciate this in the case of the standard complex Pauli matrices remember the matrix representation of the complex numbers.

$$(a+ib)^* \quad = \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{\mathsf{T}} \quad = \quad (a-ib) \tag{75}$$

The real part a is symmetric while the imaginary part b is anti symmetric. Also note that the imaginary Pauli matrix becomes symmetric when we substitute the i by their matrix representation.

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{76}$$

Which is why all the real valued Pauli matrices are symmetric.

2.18 The single column real spinors

We use 4×1 single column real valued matrices to represent spinors. Applying the basic conversion rule,

$$(a + ib) \quad = \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{77}$$

would give us 4×2 matrices. We can however avoid this if we considder that the second column doesn't contain extra information an can be derived in a simple way from the first one.

$$\begin{pmatrix} -b \\ a \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{78}$$

The matrix which does this here is just the number i. The two columns C_1 and C_2 contain the same information while they are orthogonal in the sense that $C_1^{\mathsf{T}} C_2 = 0$. Now for a contraction (complex conjugate product) $\alpha^\dagger\beta$ of two independent complex numbers we can write.

$$\begin{aligned} Re\Big\{(a+ib)^\dagger(c+id)\Big\} &= \quad \begin{pmatrix} a \\ b \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} c \\ d \end{pmatrix} &&= \quad ac + bd \\ Im\Big\{(a+ib)^\dagger(c+id)\Big\} &= \quad -\begin{pmatrix} a \\ b \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} &&= \quad ad - bc \end{aligned} \tag{79}$$

This means that we have two expressions which give us the real and imaginary parts of the contraction independently.

In the real valued spinor representation something very similar happens. The second column can be derived from the first one via a multiplication with the matrix **i**.

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{pmatrix} = \mathbf{i}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{80}$$

The second column is just the first column multiplied by **i**. In the general case an expression like $\psi_1^\dagger\,\psi_2$ will produce a complex result with both a real and an imaginary part which can be obtained independently in the same way with.

$$\begin{aligned} Re\Big\{\,\psi_1^\dagger\psi_2\,\Big\} &\longrightarrow \qquad \psi_1^\mathsf{T}\psi_2 \\ Im\Big\{\,\psi_1^\dagger\psi_2\,\Big\} &\longrightarrow \quad -\,\psi_1^\mathsf{T}\mathbf{i}\psi_2 \end{aligned} \tag{81}$$

2.19 The transition interference current

The expression for the vector current $\bar{\psi}\gamma^\mu\psi$ is per definition real, even in the case of the interference current of a transition going from an initial state ψ_i to a final state ψ_f.

$$\overline{(\psi_i+\psi_f)}\,\gamma^\mu\,(\psi_i+\psi_f) \;=\; \cdots \;+\; \bar{\psi}_i\gamma^\mu\psi_f \;+\; \bar{\psi}_f\gamma^\mu\psi_i \;+\; \cdots \tag{82}$$

The interference part of the current is given by the two complementary terms on the righthand side which together are real valued. Nevertheless, in a typical Feynman diagram only one of them is used like in $\bar{u}_f\,\gamma^\mu\,u_i$ We thus have to consider both the real and the imaginary part.

But why is this the case? Both complementary terms contain the same information and each term has the information required to do the calculations so that's not the problem. The point is: Why do we need to consider the imaginary part also and what does 'imaginary' mean here in our real valued representation where we only should need to consider contractions with a single real valued number as a result.

Averaging over phase

The reason is that the phase relation between the incoming and the outgoing state is not specifically defined in a Feynman diagram and in fact we want to average over all possible phase relations rather then specify a certain phase relation.

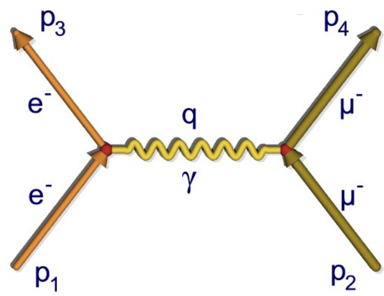

Figure 14: Mott scattering (electron - muon)

A change in the phase relation, for instance between the spinors u_1 and u_3 belonging to the incoming and outgoing momenta p_1 and p_3 in figure (14) causes a shift in the real valued interference current pattern. If we shift one spinor by -90^o degrees as in $u_3 \rightarrow -\mathbf{i}u_3$ then the (real valued) interference pattern shifts by -90^o as well since p_3 is defined by $u_3 \exp(-\mathbf{i}\, p_i x^i)$

$$\begin{array}{rclcl} \mathsf{Re}\big(j_{tr}\big) & = & \bar{u}_1\gamma^\mu\, u_3 & \rightarrow & a\, \cos{(p_3-p_1)t} \\ \mathsf{Im}\big(j_{tr}\big) & = & -\bar{u}_1\gamma^\mu \mathbf{i} u_3 & \rightarrow & b\, \sin{(p_3-p_1)t} \end{array} \tag{83}$$

So now we see that the real part and the 'imaginary' part of the transition current correspond with two orthogonal sinusoidal interference patterns. We don't need to care about the exact phase relation between u_1 and u_3 in this way. When we calculate the probability $\mathcal{M}^2$ at the end then there is no more phase dependency.

$$\mathcal{M}^2 \;=\; \mathcal{M}^*\mathcal{M} \;\propto\; a^2+b^2 \tag{84}$$

2.20 The bilinear fields

The Dirac field bilinears are defined by using real valued dot products.

$$
\begin{array}{rclc}
\bar{\psi}\,\psi & = & \psi_R^{\mathsf{T}}\,\psi_R \;+\; \psi_L^{\mathsf{T}}\,\psi_L & \text{scalar} \\
\bar{\psi}\,\gamma^\mu\,\psi & = & \psi_R^{\mathsf{T}}\,\sigma^\mu\,\psi_R \;+\; \psi_L^{\mathsf{T}}\,\tilde{\sigma}^\mu\,\psi_L & \text{vector} \\
\bar{\psi}\,\gamma^\mu\gamma^5\,\psi & = & \psi_R^{\mathsf{T}}\,\sigma^\mu\,\psi_R \;-\; \psi_L^{\mathsf{T}}\,\tilde{\sigma}^\mu\,\psi_L & \text{axial vector} \\
\bar{\psi}\,\mathbf{i}\,\gamma^5\,\psi & = & \psi_R^{\mathsf{T}}\,\mathbf{i}\,\psi_R \;-\; \psi_L^{\mathsf{T}}\,\mathbf{i}\,\psi_L & \text{pseudo scalar}
\end{array}
\qquad (85)
$$

These are all real valued expressions. For the tensor we have two expressions, one for the real part (the magnetization components) and one for the imaginary part (the polarization components). These components are zero in the tensor in which they aren't defined.

$$
\begin{array}{cc}
\bar{\psi}\,\sigma^{\mu\nu}\,\psi & \text{tensor (magnetization)} \\
\bar{\psi}\,\mathbf{i}\,\sigma^{\mu\nu}\,\psi & \text{tensor (polarization)}
\end{array}
\qquad (86)
$$

2.21 The trace theorems

The trace rules generally come to play when the spin sums are applied in calculating Feynman diagrams. The trace rules for the 8×8 gamma matrices are the just the same as the usual ones, with one notation to make: The trace of an 8×8 unit matrix is $\mathrm{Tr}(\mathbf{1}) = 8$ instead of 4 in the case of the 4×4 complex ones.

Where the standard complex calculations use a factor $\frac{1}{4}$ in order to use the spin *average* instead of the spin sum itself we have to apply a factor $\frac{1}{8}$ to do the same, for example.

Trace theorems for 8×8 real valued gamma matrices

$$
\begin{array}{lcl}
\mathrm{Tr}(\mathbf{1}) & = & 8 \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) & = & 8\; g^{\mu\nu} \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) & = & 8\,(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})
\end{array}
\qquad (87)
$$

2.22 The spin sum rules

The spin sum rule, for calculations where the spin is averaged out, and accompanying trace technology can be used equally well in the case of the real valued Dirac equation.

The two spin states for a certain direction are given by the eigenvectors of the boost generator in that direction. We did see that with 4×4 real matrices we get two orthogonal eigenvectors for each spin state, up and down so that we can write.

$$\xi(t) = \xi_1 \cos(\omega t) - \xi_2 \sin(\omega t) \qquad \text{versus} \qquad \xi(t) = \xi\, e^{-i\omega t} \tag{88}$$

The degenerate pair of eigenvectors makes it possible that the spin can spin. Each value of $\xi(t)$ is again an eigenvector. This means that we have to sum the spin states like this.

$$\sum_{s\uparrow\ s\downarrow} u^s\, \bar{u}^s \quad \Rightarrow \quad \sum_{s\uparrow_1\ s\uparrow_2\ s\downarrow_1\ s\downarrow_2} u^s\, \bar{u}^s \tag{89}$$

Over all four eigenvectors. The eigenvectors we get from the Pauli matrices represent the spinor states in the rest frame. In the rest frame the spin sum is given by.

$$\sum_{spin} u^s\, \bar{u}^s \ = \ \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \tag{90}$$

Due to the orthogonality of the eigenvectors and the fact that the eigenvalues are +1 (up) and -1 (down). For the general sum rule we must boost the chiral spinor to an arbitrary frame. The left and right 2-spinors ξ are equal in the rest frame. After a boost they become different

$$\psi \ = \ \begin{pmatrix} \xi \\ \xi \end{pmatrix} e^{-\mathbf{i}E_o t} \ \ \overset{\text{boost}}{\Rightarrow} \ \ \begin{pmatrix} \xi'_L \\ \xi'_R \end{pmatrix} e^{-\mathbf{i}Et+\mathbf{i}\vec{p}\cdot\vec{x}} \tag{91}$$

The general boost operator uses the generators in the exponentials.

$$\begin{pmatrix} \psi_L \\ \\ \psi_R \end{pmatrix} \ \ \overset{\text{boost}}{\Rightarrow} \ \ \begin{pmatrix} \exp\left(-\frac{1}{2}\vartheta_i\sigma^i\right)\ \psi_L \\ \\ \exp\left(+\frac{1}{2}\vartheta_i\sigma^i\right)\ \psi_R \end{pmatrix} \tag{92}$$

The square of the Pauli matrices is -1 which means that we can split the Taylor expansion of the exponential function into a cosh and a sinh part with no more matrices in the arguments. The factor $\frac{1}{2}$ in the argument becomes an overall square root.

$$\exp\left(\pm\tfrac{1}{2}\vartheta_i\sigma^i\right) \;=\; \sqrt{\mathbf{1}\cosh\vartheta \;\pm\; \vartheta_i\sigma^i\sinh\vartheta} \tag{93}$$

So that we can write the general boost operator in a compact form.

$$\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} e^{-\mathrm{i}E_o t} \quad \overset{\text{boost}}{\Rightarrow} \quad \begin{pmatrix}\sqrt{p_\mu\ \sigma^\mu}\ \ \xi\\ \sqrt{p^\mu\ \sigma^\mu}\ \ \xi\end{pmatrix} e^{-\mathrm{i}Et+\mathrm{i}\vec{p}\cdot\vec{x}} \tag{94}$$

We can thus write for the bi-spinors of the plane-wave eigen-function for the Dirac particle and its anti-particle.

$$u(p) \;=\; \begin{pmatrix}\sqrt{p_\mu\ \sigma^\mu}\ \ \xi\\ \sqrt{p^\mu\ \sigma^\mu}\ \ \xi\end{pmatrix}, \qquad v(p) \;=\; \begin{pmatrix}+\sqrt{p_\mu\ \sigma^\mu}\ \ \xi\\ -\sqrt{p^\mu\ \sigma^\mu}\ \ \xi\end{pmatrix} \tag{95}$$

Where the sign of the right chiral component changes because the coupling between the two components changes from m to -m. We can now calculate the extra factors from the boost for the spin sum rule.

$$\sum_{spin} u^s\,\bar{u}^s \;=\; \begin{pmatrix}\sqrt{p_\mu\ \sigma^\mu}\sqrt{p^\mu\ \sigma^\mu} & \sqrt{p_\mu\ \sigma^\mu}\sqrt{p_\mu\ \sigma^\mu}\\ \sqrt{p^\mu\ \sigma^\mu}\sqrt{p^\mu\ \sigma^\mu} & \sqrt{p^\mu\ \sigma^\mu}\sqrt{p_\mu\ \sigma^\mu}\end{pmatrix} \tag{96}$$

Spin sum rules for the electron and positron

$$\begin{aligned}
\text{electron:}\quad & \sum_{spin} u^s\,\bar{u}^s \;=\; \begin{pmatrix} m & p_\mu\sigma^\mu\\ p^\mu\sigma^\mu & m\end{pmatrix} \;=\; \gamma^\mu p_\mu + m\\
\text{positron:}\quad & \sum_{spin} v^s\,\bar{v}^s \;=\; \begin{pmatrix} -m & p_\mu\sigma^\mu\\ p^\mu\sigma^\mu & -m\end{pmatrix} \;=\; \gamma^\mu p_\mu - m
\end{aligned} \tag{97}$$

Nothing on the last page is specific to the fact that our gamma matrices are now real valued 8×8 matrices. It all follows from the commutation rules and the Lie algebra of the Lorentz group.

2.23 The gauge invariant Lagrangian

The gauge invariant Lagrangian will be Abelian as long as $\mathbf{i}$ is a globally fixed (unitary) linear combination of $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$. Such that,

$$\mathbf{i} \quad = \quad c_x\mathbf{i}_x + c_y\mathbf{i}_y + c_z\mathbf{i}_z \tag{98}$$

For any such combination the Lagrangian acts as the usual Abelian QED Lagrangian. If we allow the c_i to vary locally and we require the Lagrangian to posses local gauge invariance then it becomes non-Abelian and extra field components and currents are introduced.

2.24 The Abelian Lagrangian

Complex dot products such as $\psi_L^\dagger\psi_L$ are simplified to real valued dot products $\psi_L^\intercal\psi_L$. This is obvious in the case that both terms are identical since.

$$\psi_L^\intercal\psi_L \quad = \quad \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 \tag{99}$$

Since $\bar{\psi}$ is now just $\psi\gamma^o$ without the complex conjugate we can simply take the derivative of both $\bar{\psi}$ and ψ with regard to ψ itself in the Euler Lagrange procedure to obtain the equation of motion. For the Lagrangian this means the same factor $\frac{1}{2}$ as in the real valued Klein Gordon equation for the same reason.

$$\mathcal{L}_{QED} \quad = \quad \tfrac{1}{2}\,\bar{\psi}\,\mathbf{i}\,\gamma^\mu\left(\partial_\mu + \mathbf{i}eA_\mu\right)\psi \;-\; \tfrac{1}{4}F^\mu{}_\nu F^\mu{}_\nu \;-\; \tfrac{1}{2}m\,\bar{\psi}\psi \tag{100}$$

Gives us the real valued, symmetric Dirac equation.

$$\mathbf{i}\gamma^\mu\left(\partial_\mu + \mathbf{i}eA_\mu\right)\psi \quad = \quad m\psi \tag{101}$$

As well as the equation for the vector field A^μ which arises due to the local gauge invariance of the Abelian U(1) phase.

$$-\partial^\nu F^\mu{}_\nu \quad = \quad \bar{\psi}\gamma^\mu\psi \tag{102}$$

This tells us that the current density $j^\mu = \bar{\psi}\gamma^\mu\psi$ acts as the source of the (massless) vector field A^μ, the electromagnetic field of QED. Where A^μ is responsible for the Abelian U(1) phase changes along the x^μ directions and where $F^\mu_{\ \nu}$ is the field tensor,

$$F^\mu_{\ \nu} \;=\; \partial^\mu A_\nu - \partial_\nu A^\mu \tag{103}$$

expressed in Lorentz group generator form with J and K as the generators of rotations and boosts for (axial-)vector fields.

$$F^\mu_{\ \nu} \;=\; \mathsf{E}\cdot\mathsf{K} \;-\; \mathsf{B}\cdot\mathsf{J} \;=\; \begin{pmatrix} 0 & \mathsf{E}_x & \mathsf{E}_y & \mathsf{E}_z \\ \mathsf{E}_x & 0 & \mathsf{B}_z & -\mathsf{B}_y \\ \mathsf{E}_y & -\mathsf{B}_z & 0 & \mathsf{B}_x \\ \mathsf{E}_z & \mathsf{B}_y & -\mathsf{B}_x & 0 \end{pmatrix} \tag{104}$$

So that the change of the (axial-)current in proper time is given by.

$$\begin{aligned} \frac{\partial}{\partial\tau}\Big(\bar{\psi}\gamma^\mu\psi\Big) &= \tfrac{q}{m}\, F^\mu_{\ \nu}\Big(\bar{\psi}\gamma^\nu\psi\Big) \\ \frac{\partial}{\partial\tau}\Big(\bar{\psi}\gamma^5\gamma^\mu\psi\Big) &= \tfrac{q}{m}\, F^\mu_{\ \nu}\Big(\bar{\psi}\gamma^5\gamma^\nu\psi\Big) \end{aligned} \tag{105}$$

The equivalent Lorentz generator $\mathcal{F}^\mu_{\ \nu}$ for chiral spinors is.

$$\mathcal{F}^\mu_{\ \nu} \;=\; -\,\mathsf{B}\cdot J \;+\; \mathsf{E}\cdot K \;=\; -\,\tfrac{1}{2}\begin{pmatrix} (\mathsf{B}^i\mathbf{i} + \mathsf{E}^i)\sigma_i & 0 \\ 0 & (\mathsf{B}^i\mathbf{i} - \mathsf{E}^i)\sigma_i \end{pmatrix} \tag{106}$$

Let ψ_τ represent the chiral spinor field ψ_o after the corresponding operator has acted on it for a total (proper) time τ.

$$\psi_{(\tau)} \;=\; \exp\left(\tau\tfrac{q}{m}\,\mathcal{F}^\mu_{\ \nu}\right)\psi_o \tag{107}$$

Then we can write for the time evolution due to the field tensors.

$$\begin{aligned} \exp\left(\tau\tfrac{q}{m}\,F^\mu_{\ \nu}\right)\ \bar{\psi}_o\gamma^\nu\psi_o &= \bar{\psi}_{(\tau)}\gamma^\nu\psi_{(\tau)} \\ \exp\left(\tau\tfrac{q}{m}\,F^\mu_{\ \nu}\right)\ \bar{\psi}_o\gamma^5\gamma^\nu\psi_o &= \bar{\psi}_{(\tau)}\gamma^5\gamma^\nu\psi_{(\tau)} \end{aligned} \tag{108}$$

2.25 The eigenvectors of the free rest-frame Hamiltonian

In the local restframe and without interaction the Hamiltonian given by the Dirac equation is simplified to.

$$\tilde{H} \;=\; \frac{\partial}{\partial t} \;=\; -\begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} m \tag{109}$$

This Hamiltonian has 4 pairs of eigenvectors. Each pair contains an eigenvector with a $+im$ and a $-im$ eigenvalue. This means that we can construct oscillating 'eigenstates' with a frequency determined by the mass. The eigenvalues correspond to $\exp(imt)$ and $\exp(-imt)$ time evolution functions which, multiplied with their corresponding eigenvectors and added, produce real valued sine and cosine functions.

The four elementary eigenstates are directly related to the four diagonal axes through the center of the cube of chiral spinor parameters. These axis connect ψ_{L1} with ψ_{R1}, ψ_{L2} with ψ_{R2} and so on. We can therefor enumerate them using their correspondence to ψ_1 through ψ_4

$$\mathrm{H}_c^1 = \begin{pmatrix} \cos\phi_1 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ b\sin\phi_1 \\ b\sin\phi_1 \\ b\sin\phi_1 \end{pmatrix},\ \mathrm{H}_c^2 = \begin{pmatrix} 0 \\ \cos\phi_2 \\ 0 \\ 0 \\ \\ -b\sin\phi_2 \\ 0 \\ b\sin\phi_2 \\ -b\sin\phi_2 \end{pmatrix},\ \mathrm{H}_c^3 = \begin{pmatrix} 0 \\ 0 \\ \cos\phi_3 \\ 0 \\ \\ -b\sin\phi_3 \\ -b\sin\phi_3 \\ 0 \\ b\sin\phi_3 \end{pmatrix},\ \mathrm{H}_c^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos\phi_4 \\ \\ -b\sin\phi_4 \\ b\sin\phi_4 \\ -b\sin\phi_4 \\ 0 \end{pmatrix} \tag{110}$$

The four phases ϕ_1 to ϕ_4 are in principle independent, as well as the amplitudes. The time evolution term mt is suppressed in the sine and cosines for simplicity. The value of b is $\sqrt{1/3}$

The phase relation between the left and right chiral components is $\pm 90^o$. When the states are added however then, at certain quantized linear combinations, charged states occur. We will require that the bilinear fields of physical states do not depend on the phase ϕ.

We can readily check the correspondence to the four axis though the center of the cube by calculating their axial currents $\bar{\psi}\gamma^\mu\gamma^5\psi$. We find for the direction of the axial currents[6] in the restframe.

$$\dot{\mathrm{j}}^1_{\circlearrowleft} = \begin{pmatrix} \cos 2\phi \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \quad \dot{\mathrm{j}}^2_{\circlearrowleft} = \begin{pmatrix} \cos 2\phi \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \dot{\mathrm{j}}^3_{\circlearrowleft} = \begin{pmatrix} \cos 2\phi \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \dot{\mathrm{j}}^4_{\circlearrowleft} = \begin{pmatrix} \cos 2\phi \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{111}$$

We can rotate the H_c^j from (110) through an angle of 90^o which gives us the four H_s^j given by (112)

$$\mathrm{H}_s^1 = \begin{pmatrix} -\sin\phi_1 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ b\cos\phi_1 \\ b\cos\phi_1 \\ b\cos\phi_1 \end{pmatrix}, \ \mathrm{H}_s^2 = \begin{pmatrix} 0 \\ -\sin\phi_2 \\ 0 \\ 0 \\ \\ -b\cos\phi_2 \\ 0 \\ b\cos\phi_2 \\ -b\cos\phi_2 \end{pmatrix}, \ \mathrm{H}_s^3 = \begin{pmatrix} 0 \\ 0 \\ -\sin\phi_3 \\ 0 \\ \\ -b\cos\phi_3 \\ -b\cos\phi_3 \\ 0 \\ b\cos\phi_3 \end{pmatrix}, \ \mathrm{H}_s^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin\phi_4 \\ \\ -b\cos\phi_4 \\ b\cos\phi_4 \\ -b\cos\phi_4 \\ 0 \end{pmatrix} \tag{112}$$

We can use these sets to express any particle state with $\kappa = 1..4$ as.

$$c^\kappa H_c^\kappa \ + \ s^\kappa H_s^\kappa \qquad = \qquad \left(c^\kappa + \begin{pmatrix} 0 & \mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} s^\kappa \right) H_c^\kappa \tag{113}$$

2.26 Allowed particle states with phase independent bilinears

If we require that there is no dependency on the phase ϕ in any of the five types of bilinear fields then we get eight equations in the eight real parameters c^e and s^e. The shortest way to write these eight equations is.

$$(c^\kappa + is^\kappa)\ \sigma^\kappa\ (c^\kappa + is^\kappa) = 0 \qquad \Rightarrow \qquad a^\kappa\ \sigma^\kappa\ a^\kappa = 0 \tag{114}$$

[6] Note the dependence on the phase ϕ which has to be eliminated for physical states.

Where each of the four real $\sigma^\kappa = \sigma^\mu$ is responsible for two equations, one for the real and one for the imaginary part. However, this reduces to just four equations in four complex variables with the complex a^e and these equations can be solved exactly.

$$\left\{ \begin{array}{rcl} & a_1 & \\ & a_2 & \\ a_3 & = & +\,a_1\,e^{\pm\frac{1}{3}i\pi} - a_2\,e^{\mp\frac{1}{3}i\pi} \\ a_4 & = & -\,a_1\,e^{\mp\frac{1}{3}i\pi} - a_2\,e^{\pm\frac{1}{3}i\pi} \end{array} \right\} \tag{115}$$

Which leaves us with two complex variables a_1 and a_2 which can be chosen independently. For each combination of a_1 and a_2 there are just two sets of a_3 and a_4 which correspond with the electron and the positron. (All signs flip simultaneously). The possible particle states are given by.

$$\psi \quad = \quad \left(\mathsf{Re}(a^\kappa) + \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \mathsf{Im}(a^\kappa) \right) H_c^\kappa \tag{116}$$

The allowed spin up and spin down states states for each x^i is given by.

$$\text{spin up:}\;\; \gamma^i\,\psi = -\gamma^5\,\psi, \qquad \text{spin down:}\;\; \gamma^i\,\psi = +\gamma^5\,\psi \tag{117}$$

2.27 The electron and positron from the eigenstates

We find that the spin sum rule (the completeness rule) of the four Hamiltonian eigenstates and the alternative set is time-dependent. We get the following spin sum for the H_c^κ from (110).

$$\sum_{\kappa=1..4} H_c^\kappa\;\bar{H}_c^\kappa \;=\frac{1}{2}\begin{pmatrix} -\mathbf{i}m\,\sin 2\phi t & p_\mu\,\sigma^\mu(1+\cos 2\phi t) \\ p^\mu\,\sigma^\mu(1-\cos 2\phi t) & +\mathbf{i}m\,\sin 2\phi t \end{pmatrix} \tag{118}$$

There shouldn't be any dependency on the phase ϕ for stable particle states. To see how we can use the Hamiltonian eigenstates to construct electron and positron eigenstates we first define the set of eigenstates corresponding with the parameters of the other chiral state. We can do so due to the degeneracies in the eigenvalues.

These eigenstates have the left and right chiral components swapped compared to the those of the H_c from (110).

$$\mathrm{H}_c^{'1} = \begin{pmatrix} 0 \\ b\sin\phi_1 \\ b\sin\phi_1 \\ b\sin\phi_1 \\ \\ \cos\phi_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathrm{H}_c^{'2} = \begin{pmatrix} -b\sin\phi_2 \\ 0 \\ b\sin\phi_2 \\ -b\sin\phi_2 \\ \\ 0 \\ \cos\phi_2 \\ 0 \\ 0 \end{pmatrix}, \mathrm{H}_c^{'3} = \begin{pmatrix} -b\sin\phi_3 \\ -b\sin\phi_3 \\ 0 \\ b\sin\phi_3 \\ \\ 0 \\ 0 \\ \cos\phi_3 \\ 0 \end{pmatrix}, \mathrm{H}_c^{'4} = \begin{pmatrix} -b\sin\phi_4 \\ b\sin\phi_4 \\ -b\sin\phi_4 \\ 0 \\ \\ 0 \\ 0 \\ 0 \\ \cos\phi_4 \end{pmatrix} \tag{119}$$

The matrix of hamiltonian time-evolution generator $\tilde{H}$ is used to induce the 90^o phase shift with regard to H_c^κ. We can now define two new sets of states using H_c^κ and $H_c'^\kappa$.

$$\begin{aligned} H_-^\kappa &= H_c^\kappa + H_c'^\kappa \quad \text{electron} \\ H_+^\kappa &= H_c^\kappa - H_c'^\kappa \quad \text{positron} \end{aligned} \tag{120}$$

The H_-^κ states are those of the electron in the restframe while the states of H_+^κ have a 180^o phase difference as in the positron case.

$$\begin{aligned} \sum_{\kappa=1..4} H_-^\kappa\, \bar{H}_-^\kappa &= \begin{pmatrix} +m & p_\mu\,\sigma^\mu \\ p^\mu\,\sigma^\mu & +m \end{pmatrix} &= \gamma^\mu p_\mu + m \\ \sum_{\kappa=1..4} H_+^\kappa\, \bar{H}_+^\kappa &= \begin{pmatrix} -m & p_\mu\,\sigma^\mu \\ p^\mu\,\sigma^\mu & -m \end{pmatrix} &= \gamma^\mu p_\mu - m \end{aligned} \tag{121}$$

Indeed when we determine the spin sums (121), which turn out to be time independent in these cases, then we get the spin sums of the electron and positron respectively.

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3 The three charges of SO(4) ≅ Spin(3) ⊗ Spin(3)

Chapter 3

Real Symmetric Representation

3.1 The charge generators of SO(4)

A single spinor has four independent parameters. The most general transformation matrix operating on the spinor is a 4×4 real valued matrix. As a result the most complete group of unitary rotation generators is given by the group SO(4). This group decomposes into two three dimensional subgroups with generators J_i^{abs} and J_i^{rel} as.

$$SO(4) \;\cong\; Spin(3)^{abs} \otimes Spin(3)^{rel} \tag{122}$$

In order see how SO(4) does split into two spin(3) groups we first write out all six generators of rotation of SO(4) in real valued 4×4 matrix form.

The six individual rotation generators of SO(4) (123)

$$J_x^{\mathrm{lo}}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad J_y^{\mathrm{lo}}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix},\quad J_z^{\mathrm{lo}}=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$J_x^{\mathrm{hi}}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad J_y^{\mathrm{hi}}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},\quad J_z^{\mathrm{hi}}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix}$$

We have organized the rotation generators in two groups, one with the generators in the lower 3×3 matrix and one with the other three. The commutation relations are given by.

$$\begin{aligned}\left[\,\vec{J}_i^{\mathrm{lo}}\,,\,\vec{J}_j^{\mathrm{lo}}\,\right] &= \vec{J}_k^{\mathrm{lo}} &\qquad \left[\,\vec{J}_i^{\mathrm{lo}}\,,\,\vec{J}_j^{\mathrm{hi}}\,\right] &= \vec{J}_k^{\mathrm{hi}}\\ \left[\,\vec{J}_i^{\mathrm{hi}}\,,\,\vec{J}_j^{\mathrm{hi}}\,\right] &= \vec{J}_k^{\mathrm{lo}} &\qquad \left[\,\vec{J}_i^{\mathrm{hi}}\,,\,\vec{J}_j^{\mathrm{lo}}\,\right] &= \vec{J}_k^{\mathrm{hi}}\end{aligned} \tag{124}$$

We see that these two subgroups mix. They are not independent. From the commutation rules we see how we can obtain the two independent spin(3) groups via the following linear combinations.

$$\vec{J}^{\mathrm{abs}} \;=\; \tfrac{1}{2}\Big(\vec{J}^{\mathrm{lo}} + \vec{J}^{\mathrm{hi}}\Big) \qquad\qquad \vec{J}^{\mathrm{rel}} \;=\; \tfrac{1}{2}\Big(\vec{J}^{\mathrm{lo}} - \vec{J}^{\mathrm{hi}}\Big) \tag{125}$$

These two subgroups do commute. They are the independent spin(3) groups in which SO(4) decomposes.

$$\begin{aligned} \left[\,\vec{J}_i^{\,\text{abs}}\,,\,\vec{J}_j^{\,\text{abs}}\,\right] &= \vec{J}_k^{\,\text{abs}} \qquad & \left[\,\vec{J}_i^{\,\text{abs}}\,,\,\vec{J}_j^{\,\text{rel}}\,\right] &= 0 \\ \left[\,\vec{J}_i^{\,\text{rel}}\,,\,\vec{J}_j^{\,\text{rel}}\,\right] &= \vec{J}_k^{\,\text{rel}} \qquad & \left[\,\vec{J}_i^{\,\text{rel}}\,,\,\vec{J}_j^{\,\text{abs}}\,\right] &= 0 \end{aligned} \tag{126}$$

The matrices of the independent spin(3) generator groups are given by.

The three SO(4) generators of absolute rotation (127)

$$J_x^{\text{abs}}=\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad J_y^{\text{abs}}=\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\quad J_z^{\text{abs}}=\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The three SO(4) generators of relative rotation (128)

$$J_x^{\text{rel}}=\frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad J_y^{\text{rel}}=\frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\quad J_z^{\text{rel}}=\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The square of any of these matrices is **-1**, the negative unity matrix.

The first set of generators, the J_i^{abs} group is used for the generators of the three dimensional absolute rotation in space. The generators of the second set, the J_i^{rel} group, become the generators of *spinor relative* rotation.

The spinor on which they operate defines a relative coordinate system and it is in this coordinate system in which the rotation takes place. The direction of the spinor defines one of the directions but a spinor is more than a vector. A spinor can be viewed as a flagpole and can as such define a relative coordinate system, something which a vector can't.

A relative rotation around the spinor's own axis is identified with the generator of electric charge and all three spinor relative generators become charge generators. This is in fact already the case in the standard complex Pauli matrix representation. The other charge generators are associated with Majorana particles, generally without realizing the SO(4) origin.

To show this we will temporary continue the discussion is the standard complex representation even though the notation is awkward and requires a complex conjugate operator * (without transpose)

Because we have defined the J_i^{abs} group, given by $J_i^{abs} = -\frac{i}{2}\sigma_i$ in the complex notation, as the generators of absolute rotation we can define the direction $\vec{s}$ of a spinor ξ_s and with this we impose the requirement from special relativity that such a spinor ξ_s is an eigenvector of a boost in the $\vec{s}$ direction,

$$(\vec{s}\cdot\vec{\sigma})\,\xi_s \;=\; s\,\xi_s \qquad \longrightarrow \qquad e^{i\vec{s}\cdot\vec{\sigma}}\xi_s \;=\; e^{is}\xi_s \tag{129}$$

Where $(\vec{s}\cdot\vec{\sigma})$ is the boost generator. As a consequence the rotation operator $\exp(i\vec{s}\cdot\vec{\sigma})$ at the righthand side, which rotates the spinor around its own axis, becomes equivalent to the charge operator $\exp(is)$. The charge generator is part of the second group J_i^{rel} and therefor the second group as a whole becomes the group of spinor *relative* rotation generators.

They rotate a spinor relative to the spinor's own direction and orientation. These relative rotators commute with the generators of absolute rotation because they don't depend on the absolute orientation of the spinor. In the standard complex representation of the Dirac equation all six generators, with the omission of the factor $\frac{1}{2}$, are given by.

absolute rotation generators		**relative rotation generators**	
	$Spin(3)^{abs}$		$Spin(3)^{rel}$
$-i\sigma_1$	x-axis rotation	σ_2*	rotation orthogonal to ξ_s
$-i\sigma_2$	y-axis rotation	$i\sigma_2*$	rotation orthogonal to ξ_s
$-i\sigma_3$	z-axis rotation	i	rotation around ξ_s's axis

commutation rules

$$\tfrac{1}{2}\Big[-i\sigma_1\,,\,-i\sigma_2\Big] = -i\sigma_3$$
$$\tfrac{1}{2}\Big[-i\sigma_2\,,\,-i\sigma_3\Big] = -i\sigma_1$$
$$\tfrac{1}{2}\Big[-i\sigma_3\,,\,-i\sigma_1\Big] = -i\sigma^2$$

commutation rules

$$\tfrac{1}{2}\Big[\;\sigma_2*\,,\,i\sigma_2*\;\Big] = \;i$$
$$\tfrac{1}{2}\Big[i\sigma_2*\,,\;i\;\Big] = \sigma_2*$$
$$\tfrac{1}{2}\Big[\;i\;,\;\sigma_2*\;\Big] = i\sigma_2*$$

3.2 The SO(4) charge generators and Majorana particles

The notation of the other two members of the second group J_i^{rel} is somewhat awkward in the standard complex notation. It requires the use of a complex conjugate operator $*$ (without transpose) which acts to the right on the spinor field. In the 4×4 real notation this conjugation operation is just a simple 4×4 diagonal matrix.

standard complex notation		**real symmetric notation**	
$\exp(-\ \omega t \sigma_2 *)\,\psi$	Majorana	$\exp(-\mathbf{i}_1 \omega t)\,\psi$	Majorana
$\exp(-i\omega t \sigma_2 *)\,\psi$	Majorana	$\exp(-\mathbf{i}_2 \omega t)\,\psi$	Majorana
$\exp(-i\omega t\ \ \ \ )\,\psi$	Dirac	$\exp(-\mathbf{i}\ \ \omega t)\,\psi$	Dirac

These two complex conjugate generators are well know in combination with the Majorana particle and its equation. The operators rotate the spinor always perpendicular to its own direction. There are two orthogonal ways to do this as shown in figure (15) at the top and hence there are two corresponding operators.

The generator $i\sigma_2 *$ is generally known as the parity operator. It always rotates a spinor around an axis orthogonal to the spinor's own direction so that a multiplication with this generator corresponds with a 180^o rotation to the opposite direction.

Note that there is an important difference with a real parity operator, something which becomes apparent at intermediate angles between 0^o and 180^o, because the three generators of J_i^{rel} commute like a rotation group.

The crucial point is that a rotation over 180^o flips the spin but leaves the handedness unchanged. A left chiral component stays left handed. We know from atomic spectroscopy that Pauli's exclusion principle allows two electrons with opposite spin in the same state, so two *separate* left chiral components with opposite spin are allowed in the same state.

This would mean that Majorana particles with two such chiral components are allowed and these particles would always be detected as left handed, even if they have mass.

3.3 Visualization of the SO(4) charge generators

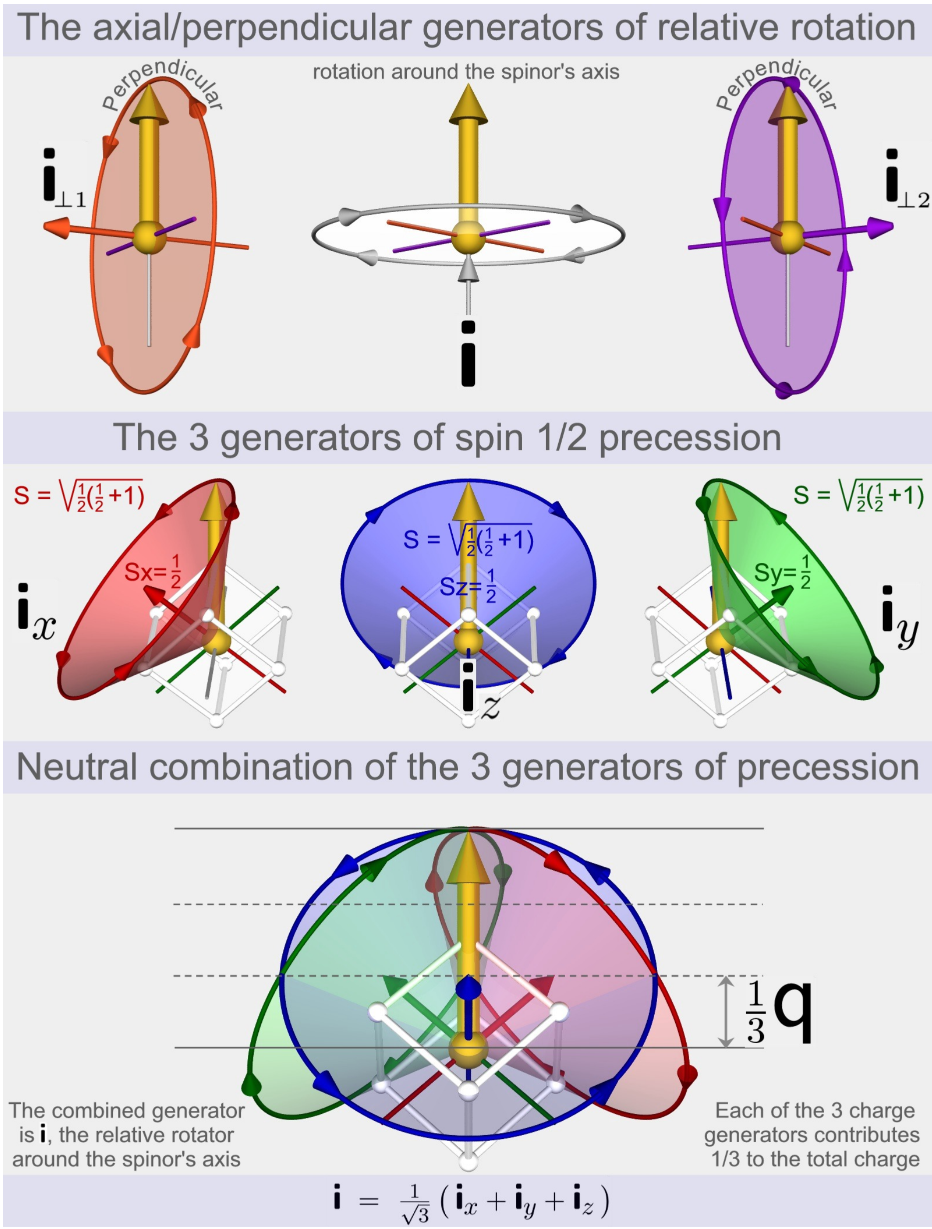

Figure 15: The choice of $\mathbf{i}$ turns the $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ into precession generators

3.4 The tribimaximal rotation matrix

The real valued representation becomes spatially symmetric when we define the generator of electric charge as.

$$\mathbf{i} \;=\; \tfrac{1}{\sqrt{3}}\left(\,\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z\,\right) \tag{130}$$

Where the matrices $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ correspond to the generators of spinor relative rotation $J_i^{rel} = \frac{1}{2}\mathbf{i}_i$. As we will see, this is not the only thing worth mentioning that happens at this particular choice. The matrix $\mathbf{i}$ becomes the generator of electric charge corresponding to rotations around the spinors own axis.

We will define the two other (perpendicular) charge generators $\mathbf{i}_1$ and $\mathbf{i}_2$ now which must perpendicular to $\mathbf{i}$. We use the so called tribimaximal rotation matrix to define the set of generators as a whole.

$$\begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i} \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{pmatrix} \begin{pmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{pmatrix} \tag{131}$$

Both sets of orthogonal generators are visualized in figure (16) which includes projection shadows in the three directions. We are free to rotate the two perpendicular rotators using an arbitrary mixing angle θ so a more general form of the rotation is given by.

$$\begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{pmatrix} \begin{pmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{pmatrix} \tag{132}$$

All three generators are orthogonal for any choice of $\cos\theta$. We will derive the set of gamma matrices for the currents associated with $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ during the treatment of the non Abelian Lagrangian, see (166). The gamma matrices for the currents generated by $\mathbf{i}_1$ and $\mathbf{i}_2$ can then be constructed using the above transformation matrix.

From the side projection in figure (16) we see that the generators $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ each contributes $\frac{1}{3}q$ to the electric charge. The top projection may remind the reader to the SU(3) root diagram.

3.5 Visualization of the Tribimaximal rotation

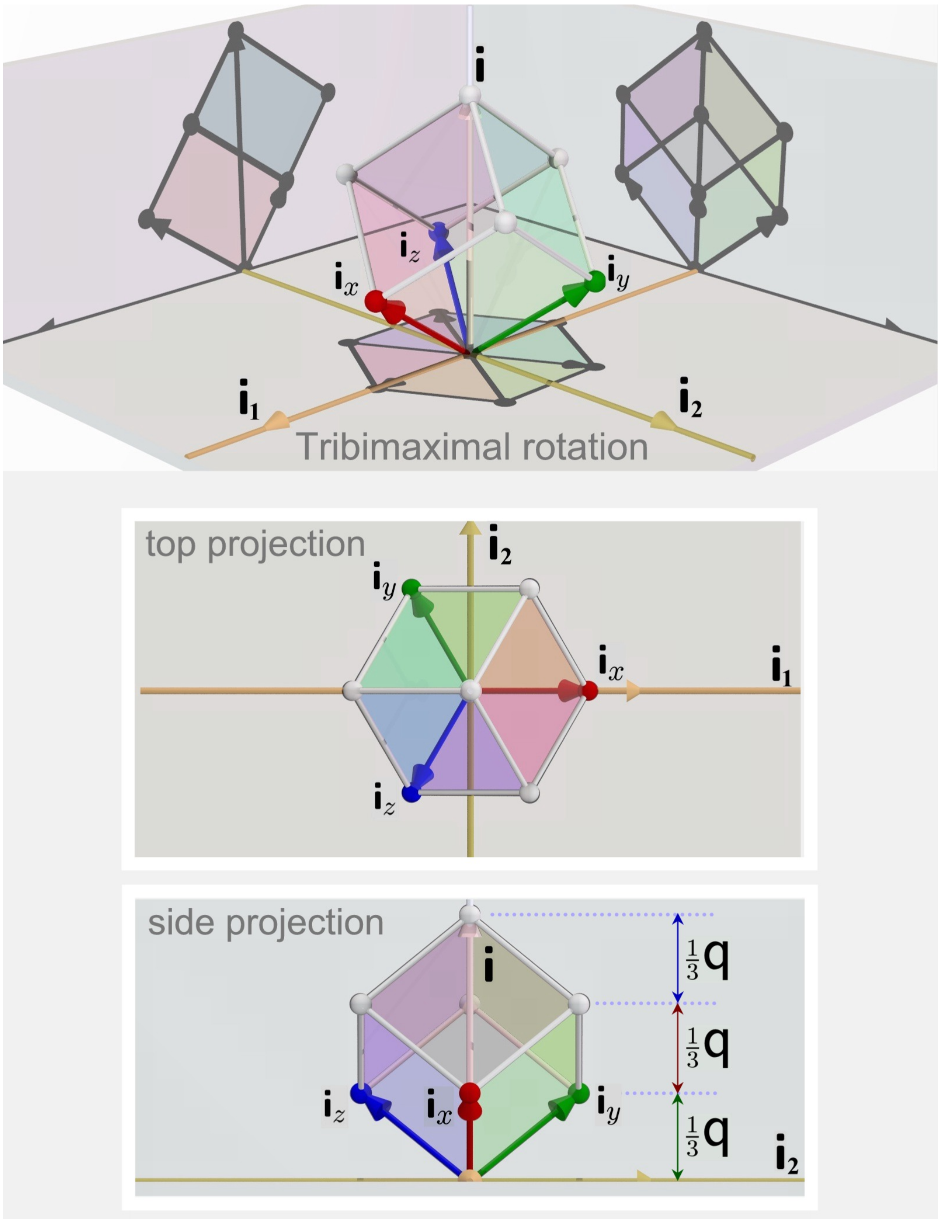

Figure 16: Symmetric SO(4) charges: Perspective, top and side view

3.6 Spinors as flagpoles and absolute rotation

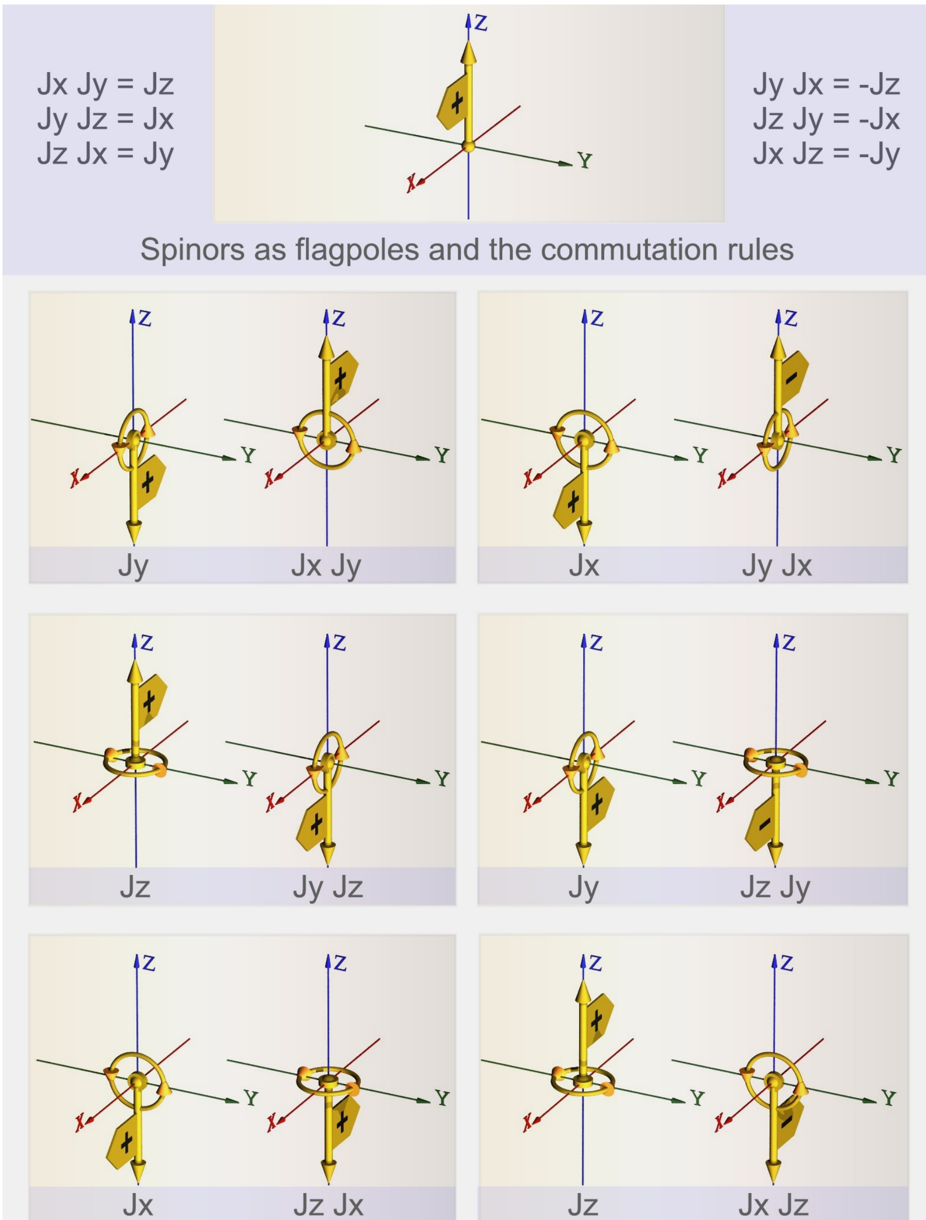

Figure 17: The six absolute rotation commutation rules visualized

3.7 The spinor relative frame and relative rotation

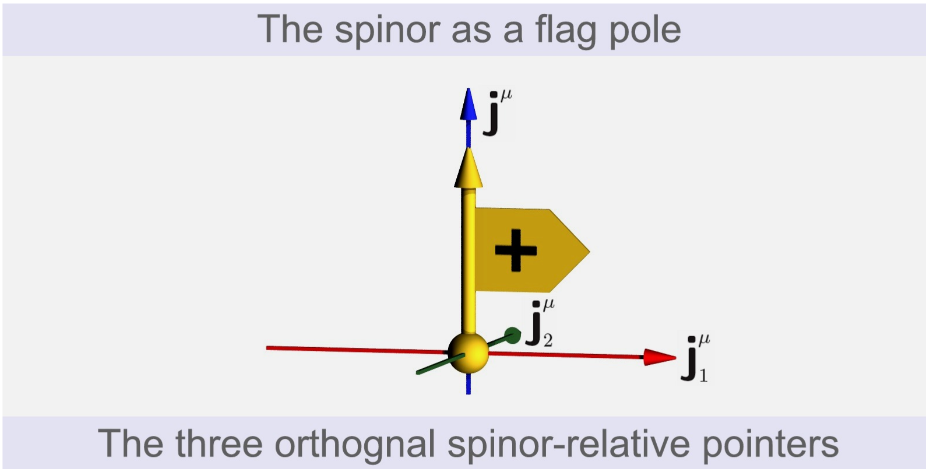

Figure 18: The non-Abelian currents as spinor relative frame pointers

Figure (17) visualizes the spinor as a flagpole under absolute rotations with regard to the x, y and z-axis. The corresponding rotation generators are from the first Spin(3) group of SO(4)

$$SO(4) \;\cong\; Spin(3)^{abs} \otimes Spin(3)^{rel} \tag{133}$$

The generators of the second Spin(3) group rotate the spinor relative to its own reference frame. This spinor reference frame is defined by three pointers (currents) defined in the real representation by.

$$\begin{aligned} \mathbf{j}^i_1 &= \psi^{\mathsf{T}} \sigma^i_1 \psi \\ \mathbf{j}^i_2 &= \psi^{\mathsf{T}} \sigma^i_2 \psi \\ \mathbf{j}^i &= \psi^{\mathsf{T}} \sigma^i \psi \end{aligned} \tag{134}$$

The last one $\mathbf{j}^i$ is just the spinor pointer obtained by using the normal Pauli matrices as applied in the vector current. The other two pointers correspond with currents which we will obtain from requiring gauge invariance of the non Abelian SO(4) Lagrangian. The corresponding sets of Pauli matrices σ^i_1 and σ^i_2 are given in (167)

3.8 The time evolution operator and the SO(4) charges

The particle states as shown in figure (15) are all possible eigenstates of the Dirac equation, however only the top tree are physically viable in the sense that they can be non-radiating. We'll look here in more detail in the possible eigenstates of the Dirac equation.

The time evolution Hamiltonian for a particle obeying the standard complex Dirac equation $i\gamma^\mu\partial_\mu\psi = m\psi$ and defined in the particle's local rest frame, so that the spatial derivatives are zero, simplifies to just.

$$H \;=\; \frac{\partial}{\partial t} \;=\; -\,i\gamma^o m \;=\; -\begin{pmatrix} 0 & im \\ im & 0 \end{pmatrix} \tag{135}$$

The time evolution operator in the particle's rest frame

$$e^{Ht} \;=\; \cos(mt)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin(mt)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{136}$$

Which can be considered as a periodically swapping operator which continuously exchanges the two chiral states.

$$e^{Ht}\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \;=\; \cos(mt)\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} - i\sin(mt)\begin{pmatrix} \xi_R \\ \xi_L \end{pmatrix} \tag{137}$$

The type of the particle is therefor determined by the relation between ψ_L and ψ_R in the local restframe. Most notably we have the electron at rest: $\psi_L = +\psi_R$ and the positron at rest: $\psi_L = -\psi_R$ which leads to the corresponding time evolutions of these particles.

The electron state:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(-imt)}\,\xi \\ e^{(-imt)}\,\xi \end{pmatrix} \tag{138}$$

The positron state:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(+imt)}\,\xi \\ -e^{(+imt)}\,\xi \end{pmatrix} \tag{139}$$

These two correspond to the two elementary plane wave states but there are many more theoretical possible states. Let us systematically go through the other possibilities. In the standard complex rotation we can relate ψ_L and ψ_R by an arbitrary phase ϕ corresponding to a 2ϕ rotation around the spinors own axis. The cases of $2\phi = 0^o$ and 360^o are the electron and positron states. The orthogonal $\pm 180^o$ states are defined by.

180^o spinor axial rotated states:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \mp\xi \\ i\,\xi \end{pmatrix} \;=\; \sqrt{2}\begin{pmatrix} \cos\left(mt \pm \frac{\pi}{4}\right)\,\xi \\ \pm\sin\left(mt \pm \frac{\pi}{4}\right)\,\xi \end{pmatrix} \tag{140}$$

We see that there is no more phase evolution $e^{i\phi}$ because the i in the ratio of the chiral components cancels with the i in the Hamiltonian generator.

The bi-spinor alternates between the left and the right chiral components. This is not a viable physical state since both the vector current as well as the axial current are alternating as a result. The alternating vector current would mean constant energy loss due to radiation.

Majorana states, up-down spinor combinations

We did see that it's the actual content of the spinor which defines the behavior of the particle. The majorana states occur when we combine a spin-up and a spin-down spinor in the general form of.

$$\psi \;=\; e^{Ht}\begin{pmatrix} \xi_\uparrow \\ e^{i\phi}\xi_\downarrow \end{pmatrix} \;=\; e^{Ht}\begin{pmatrix} \xi \\ e^{i\phi}\sigma^2 * \xi \end{pmatrix} \tag{141}$$

The extra degree of freedom ϕ comes from the different ways in which we can rotate an up-spinor into a down-spinor: any linear combination of the two perpendicular 180^o rotations via the two operators $\sigma^2 *$ and $i\sigma^2 *$. These operators form the second Spin(3) group in SO(4) together with the charge generator i. ($* =$ complex conjugate without transpose)

complex representation	real representation
$\frac{1}{2}\left[\ \sigma^2 *\, ,\ i\sigma^2 *\ \right] = \ i$	$\frac{1}{2}\left[\ \mathbf{i}_1\, ,\ \mathbf{i}_2\ \right] = \ \mathbf{i}$
$\frac{1}{2}\left[\ i\sigma^2 *\, ,\ \ i\ \ \right] = \sigma^2 *$	$\frac{1}{2}\left[\ \mathbf{i}_2\, ,\ \mathbf{i}\ \right] = \ \mathbf{i}_1$
$\frac{1}{2}\left[\ \ i\ \ ,\ \sigma^2 *\ \right] = i\sigma^2 *$	$\frac{1}{2}\left[\ \mathbf{i}\, ,\ \mathbf{i}_1\ \right] = \ \mathbf{i}_2$

The two generators of spinor relative, perpendicular rotation give rise to the two Majorana type particle states.

The Majorana states:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ \sigma^2 {*}\xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(-i\sigma^2 mt*)}\,\xi \\ e^{(-i\sigma^2 mt*)}\,\xi \end{pmatrix} \tag{142}$$

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ i\sigma^2 {*}\xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(+\,\sigma^2 mt*)}\,\xi \\ e^{(+\,\sigma^2 mt*)}\,\xi \end{pmatrix} \tag{143}$$

These two states can be combined with the use of the angle ϕ to a single Majorana state expression.

The Majorana states:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ e^{i\phi}\sigma^2 * \xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(-ie^{i\phi}\sigma^2 mt*)}\,\xi \\ e^{(-ie^{i\phi}\sigma^2 mt*)}\,\xi \end{pmatrix} \tag{144}$$

In the real symmetric representation we have the special real 4×4 matrices $\mathbf{i}_1$ and $\mathbf{i}_2$ which are the generators of spinor perpendicular rotation.

The Majorana states in the real symmetric representation:

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ \mathbf{i}_1\,\xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(-\mathbf{i}_2 mt)}\,\xi \\ e^{(-\mathbf{i}_2 mt)}\,\xi \end{pmatrix} \tag{145}$$

$$\psi(t) \;=\; e^{Ht}\begin{pmatrix} \xi \\ \mathbf{i}_2\,\xi \end{pmatrix} \;=\; \begin{pmatrix} e^{(+\mathbf{i}_1 mt)}\,\xi \\ e^{(+\mathbf{i}_1 mt)}\,\xi \end{pmatrix} \tag{146}$$

3.9 Propagation of Dirac and Majorana particles

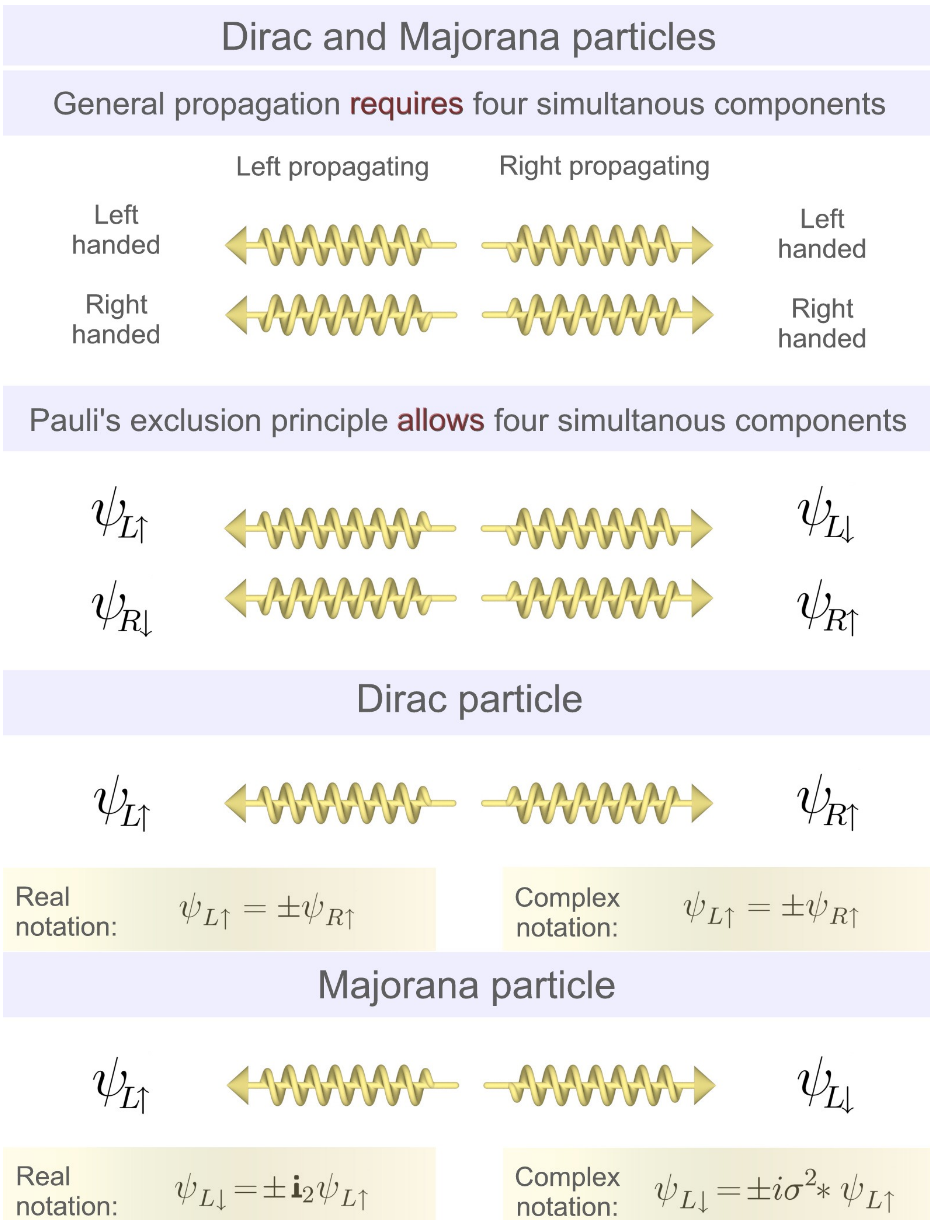

Figure 19: Dirac and Majorana particles

3.10 The currents of the SO(4) charges from the Lagrangian

In the standard complex representation there is a single current corresponding to the Abelian generator of phase i and its time evolution operator $\exp(-i\omega t)$. The Lorentz transform turns this *phase change rate in time* into a four vector j^μ depending on the pase change rates in the direction of the four coordinates x^μ.

$$i \;\longrightarrow\; \exp(-i\, p^\mu x_\mu) \;\longrightarrow\; j^\mu \;\longrightarrow\; \bar{\psi}\,\gamma^\mu\,\psi \qquad (147)$$

The gamma matrices γ^μ and the corresponding Pauli matrices come from the rotation group Spin(3) which is represented by SU(2) in the standard complex rotation.

The general transformation matrix for a single spinor (with four parameters) is a 4×4 matrix and so the complete group of unitary rotation generators is given by SO(4). This group decomposes into two Spin(3) subgroups as follows.

$$SO(4) \;\cong\; Spin(3)^{abs} \otimes Spin(3)^{rel} \qquad (148)$$

The first Spin(3) subgroup is used for the generators of the three dimensional absolute rotation in space. The second Spin(3) subgroup contains the generators of charge. This subgroup gets three currents instead of one when we impose a local gauge invariance on the group. There are two alternative ways to classify these three currents and corresponding sets of Pauli matrices.

$$\begin{array}{lllllll}
\mathbf{i}_x & \longrightarrow & \exp(-\mathbf{i}_x\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu_x & \longrightarrow & \bar{\psi}\,\gamma^\mu_{\mathsf{x}}\,\psi \\
\mathbf{i}_y & \longrightarrow & \exp(-\mathbf{i}_y\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu_y & \longrightarrow & \bar{\psi}\,\gamma^\mu_{\mathsf{y}}\,\psi \\
\mathbf{i}_z & \longrightarrow & \exp(-\mathbf{i}_z\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu_z & \longrightarrow & \bar{\psi}\,\gamma^\mu_{\mathsf{z}}\,\psi \\
 & & & & & & \\
\mathbf{i}_1 & \longrightarrow & \exp(-\mathbf{i}_1\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu_1 & \longrightarrow & \bar{\psi}\,\gamma^\mu_1\,\psi \\
\mathbf{i}_2 & \longrightarrow & \exp(-\mathbf{i}_2\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu_2 & \longrightarrow & \bar{\psi}\,\gamma^\mu_2\,\psi \\
\mathbf{i} & \longrightarrow & \exp(-\mathbf{i}\, p^\mu x_\mu) & \longrightarrow & \mathbf{j}^\mu & \longrightarrow & \bar{\psi}\,\gamma^\mu\,\psi
\end{array} \qquad (149)$$

3.11 The gauge invariant non-Abelian Lagrangian of SO(4)

We now assume that there might be conditions under which not only the phase of $\mathbf{i}$ but also the phase of $\mathbf{i}_1$ and $\mathbf{i}_2$ becomes a function of the location x^μ then the requirement of local gauge invariance of the Lagrangian will provide us with the non Abelian (connection) fields which are responsible for the variation as well as the source currents for these fields.

It is easier to start with $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$. We can obtain the Dirac equation which the corresponding non-Abelian fields directly from the QED version (37) by substituting.

$$\mathbf{i} \;=\; \tfrac{1}{\sqrt{3}}\left(\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z\right) \tag{150}$$

and adding an index to A_μ which becomes the non-Abelian A^a_μ field.

The real valued, non-Abelan Dirac equation

$$\begin{pmatrix} 0 & \mathbf{i}\sigma^\mu \\ \mathbf{i}\tilde{\sigma}^\mu & 0 \end{pmatrix}\left(\partial_\mu \;+\; \begin{pmatrix} \mathbf{i}_a & 0 \\ 0 & \mathbf{i}_a \end{pmatrix}\frac{e}{\sqrt{3}}\,\mathrm{A}^a_\mu\right)\psi \;=\; m\psi \tag{151}$$

We can express this in a more compact notation as.

$$\mathbf{i}\,\gamma^\mu D_a\psi \;=\; m\psi \tag{152}$$

Where D^a is the gauge covariant derivative which is indexed by a

From the Abelian QED Lagrangian (100) we can now write the total non-Abelian Lagrangian including the field term as.

$$\mathcal{L} \;=\; \tfrac{1}{2}\bar{\psi}\,(\mathbf{i}\gamma^\mu D^a)\,\psi \;-\; \tfrac{1}{4}\,(F^{\;\mu}_{a\,\nu})^2 \;-\; \tfrac{1}{2}m\bar{\psi}\psi \tag{153}$$

The non-Abelian field term is obtained using $[\,\mathbf{i}_a, \mathbf{i}_b] = -2\varepsilon^{abc}\,\mathbf{i}_c$

$$F^a_{\mu\nu} \;=\; \frac{[\,D^a\,,\,D^a\,]}{e\,\mathbf{i}_a\sqrt{1/3}} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{2e}{\sqrt{3}}\,\varepsilon^{abc}A^b_\mu A^c_\nu \tag{154}$$

Note that $\mathbf{i}_a^{-1}$ is just $-\mathbf{i}_a$ as in the case of the imaginary number i.

This Lagrangian leads to the Dirac equation given by varying the field ψ, whereby $\bar{\phi}$ is again just $\psi\,\gamma^o$ and ψ is the only field we need to vary in the Euler Lagrange procedure. The equations of motions for the non-Abelian fields give us the source currents $\mathbf{j}_a^\mu$ of the non-Abelian fields A_a^μ.

$$\partial^\nu F^{\,\mu}_{a\,\nu} \;-\; e'\,A^\nu_b\,F^{\,\mu}_{a\,\nu} \;=\; e'\,\bar{\psi}\left(\tfrac{1}{2}\mathbf{i}_a\gamma^\mu\right)\psi \qquad \text{with} \quad e' = \frac{2e}{\sqrt{3}} \tag{155}$$

$$j^\mu_a \;=\; -\,\bar{\psi}\left(\tfrac{1}{2}\mathbf{i}_a\gamma^\mu\right)\psi \;=\; \bar{\psi}\,\gamma^\mu_a\,\psi \tag{156}$$

The rightmost expression defines one set of gamma matrices for each of the three values of a

$$\gamma^\mu_a \;=\; \tfrac{1}{2}\begin{pmatrix} 0 & \sigma^\mu_a \\ \tilde{\sigma}^\mu_a & 0 \end{pmatrix} \tag{157}$$

Once we have the corresponding three sets of Pauli matrices for the currents corresponding with the generators $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_a$ then we can use the tribimaximal rotation to obtain the set Pauli matrices for the currents corresponding with the generators $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}$

The sets of Pauli matrices are thus related to each other by.

$$\begin{pmatrix} \sigma^\mu_1 \\ \sigma^\mu_2 \\ \sigma^\mu \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{pmatrix} \begin{pmatrix} \sigma^\mu_{\mathsf{x}} \\ \sigma^\mu_{\mathsf{y}} \\ \sigma^\mu_{\mathsf{z}} \end{pmatrix} \tag{158}$$

Where the σ^μ are the standard Pauli matrices and σ^μ_1 and σ^μ_2 are the Pauli matrices corresponding to the currents of perpendicular spinor relative rotation. The commutation relations are given by.

$$\begin{aligned} \tfrac{1}{2}\left[\,\sigma^i_{\mathsf{x}}\,,\,\sigma^j_{\mathsf{x}}\,\right] &= \mathbf{i}_x\,\sigma^k_{\mathsf{x}} & \qquad \tfrac{1}{2}\left[\,\sigma^i_1\,,\,\sigma^j_1\,\right] &= \mathbf{i}_1\,\sigma^k_1 \\ \tfrac{1}{2}\left[\,\sigma^i_{\mathsf{y}}\,,\,\sigma^j_{\mathsf{y}}\,\right] &= \mathbf{i}_y\,\sigma^k_{\mathsf{y}} & \qquad \tfrac{1}{2}\left[\,\sigma^i_2\,,\,\sigma^j_2\,\right] &= \mathbf{i}_2\,\sigma^k_2 \\ \tfrac{1}{2}\left[\,\sigma^i_{\mathsf{z}}\,,\,\sigma^j_{\mathsf{z}}\,\right] &= \mathbf{i}_z\,\sigma^k_{\mathsf{z}} & \qquad \tfrac{1}{2}\left[\,\sigma^i\,,\,\sigma^j\,\right] &= \mathbf{i}\;\sigma^k \end{aligned} \tag{159}$$

3.12 Asymmetries in the time component Pauli matrices

Typically all Pauli matrices are symmetric while the unitary generators which are all asymmetric. For instance we have.

$$(\sigma^\mu)^\mathsf{T} \;=\; \sigma^\mu, \qquad\qquad (\mathbf{i}\sigma^\mu)^\mathsf{T} \;=\; -\,\mathbf{i}\sigma^\mu \tag{160}$$

Which is identical to the following in the standard complex representation.

$$(\sigma^\mu)^\dagger \;=\; \sigma^\mu, \qquad\qquad (i\sigma^\mu)^\dagger \;=\; -\,i\sigma^\mu \tag{161}$$

This pattern is however broken in the case of the *time* component Pauli matrices for the non-Abelian currents which can be partly or whole asymmetric which amounts to a fundamental different behavior . The corresponding *space* component Pauli matrices are always symmetric for any current.

If we define an arbitrary charge generator $(\hat{\varsigma}\cdot\hat{\mathbf{i}}\,)$ via three normalized coefficients $\hat{\varsigma}$ (see the table below) and the orthogonal base generators $\hat{\mathbf{i}}$.

$$\hat{\mathbf{i}} \;=\; \{\,\mathbf{i}_x,\;\mathbf{i}_y,\;\mathbf{i}_z\,\} \;=\; J^i_{rel} \tag{162}$$

$\hat{c}_\mathsf{x}$	1	0	0	$\hat{c}_1$	$\frac{2}{\sqrt{6}}$	$\frac{-1}{\sqrt{6}}$	$\frac{-1}{\sqrt{6}}$
$\hat{c}_\mathsf{y}$	0	1	0	$\hat{c}_2$	0	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$
$\hat{c}_\mathsf{z}$	0	0	1	$\hat{c}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$

(163)

Then we can write for the Pauli matrices of such a charge generator.

$$\sigma^o_\varsigma \;=\; -\,(\hat{c}\cdot\hat{\mathbf{i}}\,)(\hat{\varsigma}\cdot\hat{\mathbf{i}}\,), \qquad\qquad \sigma^i_\varsigma \;=\; J^i_{abs}(\hat{\varsigma}\cdot\hat{\mathbf{i}}\,) \tag{164}$$

We can split the time component matrices in a symmetric and an asymmetric part determined by the angle between the coefficients $\hat{\varsigma}$ and $\hat{c}$.

$$\boxed{\sigma^o_\varsigma \;=\; (\hat{\varsigma}\cdot\hat{c})\,\mathbf{1} \;+\; (\hat{\varsigma}\times\hat{c})\cdot\hat{\mathbf{i}}} \tag{165}$$

3.13 Pauli matrices for the non Abelian currents (rep.1)

Pauli matrices for $\mathbf{j}^{\mu}_{\mathsf{x}}$	Pauli matrices for $\mathbf{j}^{\mu}_{\mathsf{y}}$	Pauli matrices for $\mathbf{j}^{\mu}_{\mathsf{z}}$
$\sigma^{o}_{\mathsf{x}} = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$ $\sigma^{x}_{\mathsf{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $\sigma^{y}_{\mathsf{x}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ $\sigma^{z}_{\mathsf{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\sigma^{o}_{\mathsf{y}} = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}$ $\sigma^{x}_{\mathsf{y}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\sigma^{y}_{\mathsf{y}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $\sigma^{z}_{\mathsf{y}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\sigma^{o}_{\mathsf{z}} = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ $\sigma^{x}_{\mathsf{z}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\sigma^{y}_{\mathsf{z}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\sigma^{z}_{\mathsf{z}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(166)

3.14 Pauli matrices for the non Abelian currents (rep.2)

Pauli matrices for $\mathbf{j}^{\mu}_{1}$	Pauli matrices for $\mathbf{j}^{\mu}_{2}$	Pauli matrices for $\mathbf{j}^{\mu}$
$\sigma^o_1 = \frac{1}{\sqrt{2}} \times \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$	$\sigma^o_2 = \frac{1}{\sqrt{6}} \times \begin{pmatrix} 0 & 2 & -1 & -1 \\ -2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & -1 & -2 & 0 \end{pmatrix}$	$\sigma^o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\sigma^x_1 = \frac{1}{\sqrt{6}} \times \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & -1 & 0 & -2 \end{pmatrix}$	$\sigma^x_2 = \frac{1}{\sqrt{2}} \times \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$	$\sigma^x = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$
$\sigma^y_1 = \frac{1}{\sqrt{6}} \times \begin{pmatrix} -1 & -1 & 0 & -2 \\ -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & -1 \\ -2 & 0 & -1 & 1 \end{pmatrix}$	$\sigma^y_2 = \frac{1}{\sqrt{2}} \times \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$	$\sigma^y = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}$
$\sigma^z_1 = \frac{1}{\sqrt{6}} \times \begin{pmatrix} -1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{pmatrix}$	$\sigma^z_2 = \frac{1}{\sqrt{2}} \times \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\sigma^z = \frac{1}{\sqrt{3}} \times \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

(167)

3.15 The general expression for the non-Abelian currents

Here we set out to calculate how the non-Abelian currents depend on a time evolution of the non-Abelian phase of a spinor in the general form of

$$\xi_s \quad \longrightarrow \quad \exp(-\tfrac{1}{2}\vec{\omega}\cdot\hat{\mathbf{i}}\,t)\;\xi_s \;=\; \psi \tag{168}$$

Where $\hat{\mathbf{i}}$ is an orthogonal set of unitary charge generators for instance.

$$\hat{\mathbf{i}} \;=\; \{\;\mathbf{i}_x,\;\mathbf{i}_y,\;\mathbf{i}_z\;\} \tag{169}$$

Or another orthogonal set as long as it obeys the commutation rules.

$$\tfrac{1}{2}[\;\mathbf{i}_x\;,\;\mathbf{i}_y\,] = \mathbf{i}_z, \qquad \tfrac{1}{2}[\;\mathbf{i}_y\;,\;\mathbf{i}_z\,] = \mathbf{i}_x, \qquad \tfrac{1}{2}[\;\mathbf{i}_z\;,\;\mathbf{i}_x\,] = \mathbf{i}_y, \tag{170}$$

In order to calculate the vector and axial currents we'll need to be able to calculate the bilinear expression for a (single) spinor.

$$\psi^{\mathsf{T}}\;\sigma^{\mu}_{\varsigma}\;\psi \;=\; \xi_s^{\mathsf{T}}\;\exp(\tfrac{1}{2}\vec{\omega}\cdot\hat{\mathbf{i}}\,t)\;\sigma^{\mu}_{\varsigma}\;\exp(-\tfrac{1}{2}\vec{\omega}\cdot\hat{\mathbf{i}}\,t)\;\xi_s \tag{171}$$

Where the Pauli matrices σ^{μ}_{ς} determine the current corresponding to the arbitrary generator $(\hat{\varsigma}\cdot\hat{\mathbf{i}}\,)$ which are given by (164) as.

$$\begin{aligned} \sigma^{o}_{\varsigma} &= (\hat{\varsigma}\cdot\hat{c})\,\mathbf{1} + (\hat{\varsigma}\times\hat{c})\cdot\hat{\mathbf{i}} \\ \sigma^{i}_{\varsigma} &= J^{i}_{abs}\,(\hat{\varsigma}\cdot\hat{\mathbf{i}}\,) \end{aligned} \tag{172}$$

The hats denote unit vectors. The quantity $(\hat{c}\cdot\hat{\mathbf{i}}\,)$ used here is the base of the representation. In our symmetric representation this base is fixed to.

$$\mathbf{i} \;=\; c_x\mathbf{i}_x + c_y\mathbf{i}_y + c_z\mathbf{i}_z \;=\; \tfrac{1}{\sqrt{3}}\big(\;\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z\big) \tag{173}$$

The same base occurs also in the expression for a unit spinor ξ_s pointing in the $\vec{s}$ direction given by. (see equation 15)

$$\xi_s \;=\; \frac{(\;s+c\;)}{\|\;s+c\;\|} \qquad \text{with:} \qquad \begin{aligned} s &= (\,0,\;s_x,\;s_y,\;s_z\,) \\ c &= (\,0,\;c_x,\;c_y,\;c_z\,) \end{aligned} \tag{174}$$

All definitions required to calculate the currents are now on the table.

We can handle this again with the use of the CBH equation related series.

$$e^X Y e^{-X} \quad = \quad Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots \tag{175}$$

The series is trival if Y commutes with X so we look first for the non-commuting parts of the general Pauli matrices in (172) which are substituted in Y. The generators of absolute rotation J^i_{abs} commute so they become just a factor which we can apply afterwards so we are left with.

$$\begin{aligned} \textit{non-commuting part of:} \quad \sigma^o_\varsigma \quad &= \quad (\hat{\varsigma} \times \hat{c}) \cdot \hat{\mathbf{i}} \\ \textit{non-commuting part of:} \quad \sigma^i_\varsigma \quad &= \quad (\hat{\varsigma} \cdot \hat{\mathbf{i}}\,) \end{aligned} \tag{176}$$

The relevant factor in X is given by $(\vec{\omega} \cdot \vec{\mathbf{i}})$ or expressed with a unit vector like $\omega(\hat{\omega} \cdot \hat{\mathbf{i}})$. For the commutation calculations we'll make us of the rule.

$$(\hat{a} \cdot \hat{\mathbf{i}})(\hat{b} \cdot \hat{\mathbf{i}}) \quad = \quad -\,(\hat{a} \cdot \hat{b})\,\mathbf{1} \;+\; (\hat{a} \times \hat{b}) \cdot \hat{\mathbf{i}} \tag{177}$$

The right most term survives in the commutation so we have.

$$\Big[\; \hat{a} \cdot \hat{\mathbf{i}} \;,\; \hat{b} \cdot \hat{\mathbf{i}} \;\Big] \quad = \quad 2\,(\hat{a} \times \hat{b}) \cdot \hat{\mathbf{i}} \tag{178}$$

So from the term $[X,Y]$ we find the non-commutative parts which will go into the rest of the series.

$$\begin{aligned} \Big[\; (\hat{\omega} \cdot \hat{\mathbf{i}}) \;,\; \sigma^o_\varsigma \;\Big] \quad &\propto \quad (\hat{\omega} \times (\hat{\varsigma} \times \hat{c})) \cdot \hat{\mathbf{i}} \\ \Big[\; (\hat{\omega} \cdot \hat{\mathbf{i}}) \;,\; \sigma^i_\varsigma \;\Big] \quad &\propto \quad (\hat{\omega} \times \hat{\varsigma}) \cdot \hat{\mathbf{i}} \end{aligned} \tag{179}$$

At this point we can see it coming that the terms $[X,Y]$ series become repeated cross products with $\hat{\omega}$ in the form of.

$$\hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times (\cdots \tag{180}$$

The results will alternate between two vectors which are both orthogonal to $\hat{\omega}$ while they are also orthogonal between the two of them.

We'll make us of the following general vector identity.

$$\hat{a} \times (\hat{b} \times \hat{c}) \quad = \quad (\hat{a} \cdot \hat{c})\, \hat{b} \; - \; (\hat{a} \cdot \hat{b})\, \hat{c} \tag{181}$$

Which simplifies in the repeated case here to.

$$(\hat{\omega} \times (\hat{\omega} \times \hat{\varsigma})) \quad = \quad (\hat{\omega} \cdot \hat{\varsigma})\, \hat{\omega} \; - \; \hat{\varsigma} \tag{182}$$

With this the series simplifies to a series with the expected alternating behavior between two orthogonal components.

$$\begin{array}{lcl} \hat{\varsigma} & \longrightarrow & +\,\hat{\varsigma} \\ \hat{\omega} \times \hat{\varsigma} & \longrightarrow & +\,\hat{\omega} \times \hat{\varsigma} \\ \hat{\omega} \times (\hat{\omega} \times \hat{\varsigma}) & \longrightarrow & -\,\hat{\varsigma} + (\hat{\omega} \cdot \hat{\varsigma})\hat{\omega} \\ \hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times \hat{\varsigma})) & \longrightarrow & -\,\hat{\omega} \times \hat{\varsigma} \\ \hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times \hat{\varsigma}))) & \longrightarrow & +\,\hat{\varsigma} - (\hat{\omega} \cdot \hat{\varsigma})\hat{\omega} \\ \hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times (\hat{\omega} \times \hat{\varsigma})))) & \longrightarrow & +\,\hat{\omega} \times \hat{\varsigma} \end{array} \tag{183}$$

The only exception is the first term ς at the start of the series before any cross product with $\hat{\omega}$ takes place. If we correct this with an additional $-(\hat{\omega} \cdot \hat{\varsigma})\hat{\omega}$ term then we can express the series as a cosine and a sine.

$$\Big[(\hat{\omega} \cdot \hat{\varsigma})\hat{\omega} \cdot \hat{\mathbf{i}}\Big] \; + \; \Big[(\hat{\varsigma} - (\hat{\omega} \cdot \hat{\varsigma})\hat{\omega}) \cdot \hat{\mathbf{i}}\Big] \cos \omega t \; + \; \Big[(\hat{\omega} \times \hat{\varsigma}) \cdot \hat{\mathbf{i}}\Big] \sin \omega t \tag{184}$$

We can recombine the cosine and sine into a pure exponential with a matrix argument as follows.

$$\Big[(\hat{\omega} \cdot \hat{\varsigma})\hat{\omega} \cdot \hat{\mathbf{i}}\Big] \; + \; \Big[((\hat{\omega} \times \hat{\varsigma}) \times \hat{\omega}) \cdot \hat{\mathbf{i}}\Big] \cos \omega t \; + \; \Big[(\hat{\omega} \times \hat{\varsigma}) \cdot \hat{\mathbf{i}}\Big] \sin \omega t \tag{185}$$

$$= \quad \Big[(\hat{\omega} \cdot \hat{\varsigma})\hat{\omega} \cdot \hat{\mathbf{i}}\Big] \; + \; \Big[((\hat{\omega} \times \hat{\varsigma}) \times \hat{\omega}) \cdot \hat{\mathbf{i}}\Big] \exp(-\vec{\omega} \cdot \hat{\mathbf{i}}\, t) \tag{186}$$

Here we also see that ratio between the first (invariant) term and the second (time varying) term depends on the angle between $\hat{\omega}$ and $\hat{\varsigma}$ since.

$$(\hat{\omega} \cdot \hat{\varsigma}) \; = \; \cos \theta, \qquad (\hat{\omega} \times \hat{\varsigma}) \; = \; \sin \theta \tag{187}$$

The difference between the two vectors is just $\hat{\varsigma}$ as can be seen in this way.

$$\left[(\hat{\omega}\cdot\hat{\varsigma})\hat{\omega}\cdot\hat{\mathbf{i}}\right] \; + \; \left[(\hat{\varsigma}-(\hat{\omega}\cdot\hat{\varsigma})\hat{\omega})\cdot\hat{\mathbf{i}}\right]\exp(-\vec{\omega}\cdot\hat{\mathbf{i}}\,t) \tag{188}$$

The time varying part of the current disappears if $\hat{\omega} = \hat{\varsigma}$. In this case the spinor is rotating around its own axis. This because $\hat{\varsigma}$ determines the base vectors and the relative rotation corresponding with a spinor rotating around its own axis.

At this point we can already see all properties of the spatial components of the general vector current for a single spinor. The current of a spinor going through a non-Abelian phase change in SO(4) according to.

$$\xi_s \quad \longrightarrow \quad \exp(-\tfrac{1}{2}\vec{\omega}\cdot\hat{\mathbf{i}}\,t)\;\xi_s \;=\; \psi \tag{189}$$

Measured in an arbitrary non-Abelian base $\hat{\varsigma}\cdot\hat{\mathbf{i}}$ has the following properties:

Basic properties of the general vector current

- The vector current precesses around and traces out a cone.
- The angle of the cone with its central axis is the same as the angle between the vectors $\hat{\omega}$ and $\hat{\varsigma}$
- The direction of the central axis of the cone is the same as the direction of the spinor when the base $\hat{\varsigma} = \hat{\omega}$. So it is $\hat{\omega}$ which determines the direction of the central axis.

3.16 The Lorentz transform of the SO(4) charge operators

All the charge generators of SO(4) commute with the rotation generators of SO(4) due to the general group structure,

$$SO(4) \;\cong\; Spin(3)^{abs} \otimes Spin(3)^{rel} \tag{190}$$

in which the two Spin(3) groups commute with each other. However, only the charge generator $\mathbf{i}$ of electric charge commutes with the boost operation due to the definition of rotations and boosts.

$$\begin{array}{rcll} \exp\left(-\frac{1}{2}\mathbf{i}\vec{\sigma}\cdot\vec{\phi}\right) & = & \exp\left(\quad \vec{J}\cdot\vec{\phi}\right) & \text{general rotation operator} \\ \exp\left(\pm\frac{1}{2}\;\vec{\sigma}\cdot\vec{\vartheta}\right) & = & \exp\left(\pm\mathbf{i}\vec{J}\cdot\vec{\vartheta}\right) & \text{general boost operator} \end{array} \tag{191}$$

The charge generator $\mathbf{i}$ commutes with itself and therefor with the general boost operator. Note that the rotation operator is actually independent of $\mathbf{i}$ while the Pauli matrices and the boost operator do dependent on the particular choice of $\mathbf{i}$.

So, while a pure rotation of a spinor around its own axis stays a pure rotation in any reference frame, this is not the case with rotations of the spinor around an axis perpendicular with itself. Consequently we have to transform first to the reference frame in which we want to rotate and transform back after the rotation. So a generator Y transforms like.

$$Y \;\longrightarrow\; \left(e^{-\vec{\vartheta}\cdot\vec{\sigma}/2}\right) Y \left(e^{\,\vec{\vartheta}\cdot\vec{\sigma}/2}\right) \;=\; \tag{192}$$

The consequences of the commutation relations are given by the following expression related to the Baker Campbell Hausdorff formula.

$$e^X Y e^{-X} \;=\; Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots \tag{193}$$

Where Y stands for the non-commuting charge generators $\mathbf{i}_1$ and $\mathbf{i}_2$ while as far as the X is concerned we only need to take the generator $\mathbf{i}$ into account.

The individual commutation relations of the charge operators are those of a rotation group.

$$[\,\mathbf{i}\ ,\ \mathbf{i}_1\,] = 2\mathbf{i}_2, \qquad [\,\mathbf{i}_1\ ,\ \mathbf{i}_2\,] = 2\mathbf{i}, \qquad [\,\mathbf{i}_2\ ,\ \mathbf{i}\ \,] = 2\mathbf{i}_1 \tag{194}$$

This means that the commutators in the series (193) in the cases of $\mathbf{i}_1$ and $\mathbf{i}_2$ become. (ignoring the 1/n! factors)

$$X = \frac{\mathbf{i}}{2},\ Y = \frac{\mathbf{i}_1}{2} \quad \longrightarrow \quad \frac{\mathbf{i}_1}{2}\Big\{\ \mathbf{i}, -\mathbf{1}, -\mathbf{i},\ \mathbf{1},\ \mathbf{i},\ \ \cdots\ \Big\} \tag{195}$$

$$X = \frac{\mathbf{i}}{2},\ Y = \frac{\mathbf{i}_2}{2} \quad \longrightarrow \quad \frac{\mathbf{i}_2}{2}\Big\{\ \mathbf{i}, -\mathbf{1}, -\mathbf{i},\ \mathbf{1},\ \mathbf{i},\ \ \cdots\ \Big\} \tag{196}$$

If we now exponentiate the generators into operators we get a cosine part, which always commutes, and a sine part which doesn't commute in case of $\mathbf{i}_1$ and $\mathbf{i}_2$. The sine term gets an extra boost factor in these cases.

$$\begin{aligned} e^{\alpha \mathbf{i}_1 t} \quad &\longrightarrow \quad \mathbf{1}\cos\alpha t\ +\ \mathbf{i}_1 \sin\alpha t \left(e^{\vec{\vartheta}\cdot\vec{\sigma}}\right) \\ e^{\alpha \mathbf{i}_2 t} \quad &\longrightarrow \quad \mathbf{1}\cos\alpha t\ +\ \mathbf{i}_2 \sin\alpha t \left(e^{\vec{\vartheta}\cdot\vec{\sigma}}\right) \\ e^{\alpha \mathbf{i}\, t} \quad &\longrightarrow \quad \mathbf{1}\cos\alpha t\ +\ \mathbf{i}\ \sin\alpha t \end{aligned} \tag{197}$$

The Lorentz transform guarantees with this boost correction that a perpendicular spinor rotation in its rest frame doesn't change the overall speed, or current, when viewed from another reference frame. The current $J = J_L + J_R$ remains constant under perpendicular rotation, see fig.(20)

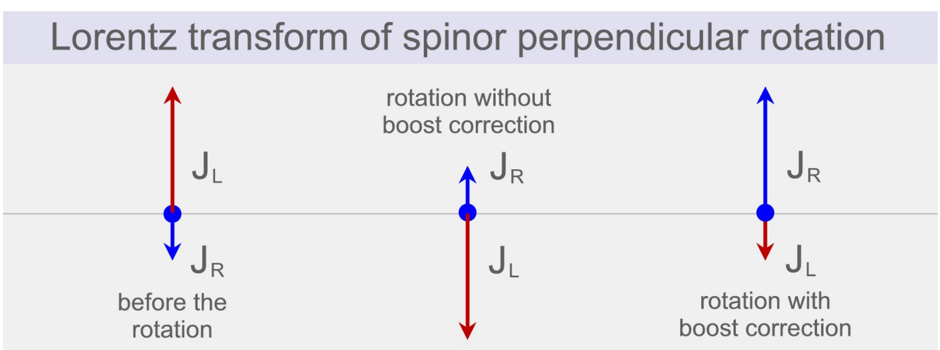

Figure 20: The current remains constant under perpendicular rotation.

3.17 An experimental hint of physical SO(4) currents

The three generators i_x, i_y and i_z become the three orthogonal generators of spin $\frac{1}{2}$ precession in the symmetric representation. By making the non-Abelian SO(4) Lagrangian locally gauge invariant we obtain the three associated current. We would like to consider these currents as physical and independently existing currents for various reasons. Here we will provide some experimental confirmation in the form of the numerical values of the electroweak charges, in other words the coupling parameters. We can have either one of the two following possibilities.

- Three generators i_x, i_y and i_z acting on a single current j^μ.
- Three generators i_x, i_y and i_z along with the three currents j^μ_x, j^μ_y, j^μ_z.

In the latter case we have $\mathsf{j}^\mu = \mathsf{j}^\mu_x + \mathsf{j}^\mu_y + \mathsf{j}^\mu_z$ and the three sub currents individually are always precessing according to the spin $\frac{1}{2}$ ratio but the total current j^μ can be anything. It can be a Dirac current with the spinor rotating around its own axis or it can be a Majorana current with the spinor rotating perpendicular to its own axis. In both cases the unwanted components cancel and only the required ones remain.

We consequently expect $\mathsf{j}^\mu = \mathsf{j}^\mu_x + \mathsf{j}^\mu_y + \mathsf{j}^\mu_z$ to be the source of the A^μ field in one configuration, as well as the source of, for example, $W^\mu_\pm$ in another configuration. However, based on the preceding paragraphs, this source process can not be 100% efficient since there are always current components that cancel. This is then where the coupling parameters come in, and they do so already at the level of the equations of motion.

We should be able to determine the efficiency of the source process from figure (21) which shows one of the three j^μ_x, j^μ_y and j^μ_z source currents. The current is precessing with an angle α which is determined by the spin $\frac{1}{2}$ of the particle.

The question is what percentage of this precessing motion translates into a rotation around the spinors own axis. From figure (21) we see that the efficiency is maximal when $\alpha = 90^o$ and the efficiency is zero when $\alpha = 0^o$. In the same way we can use the complementary angle β to determine the effective contribution of the precessing motion to a pure perpendicular rotation. It's maximum is at $\beta = 90^o$ while the efficiency is zero at $\beta = 0^o$.

3.18 Visualization of the SO(4) spinor projections

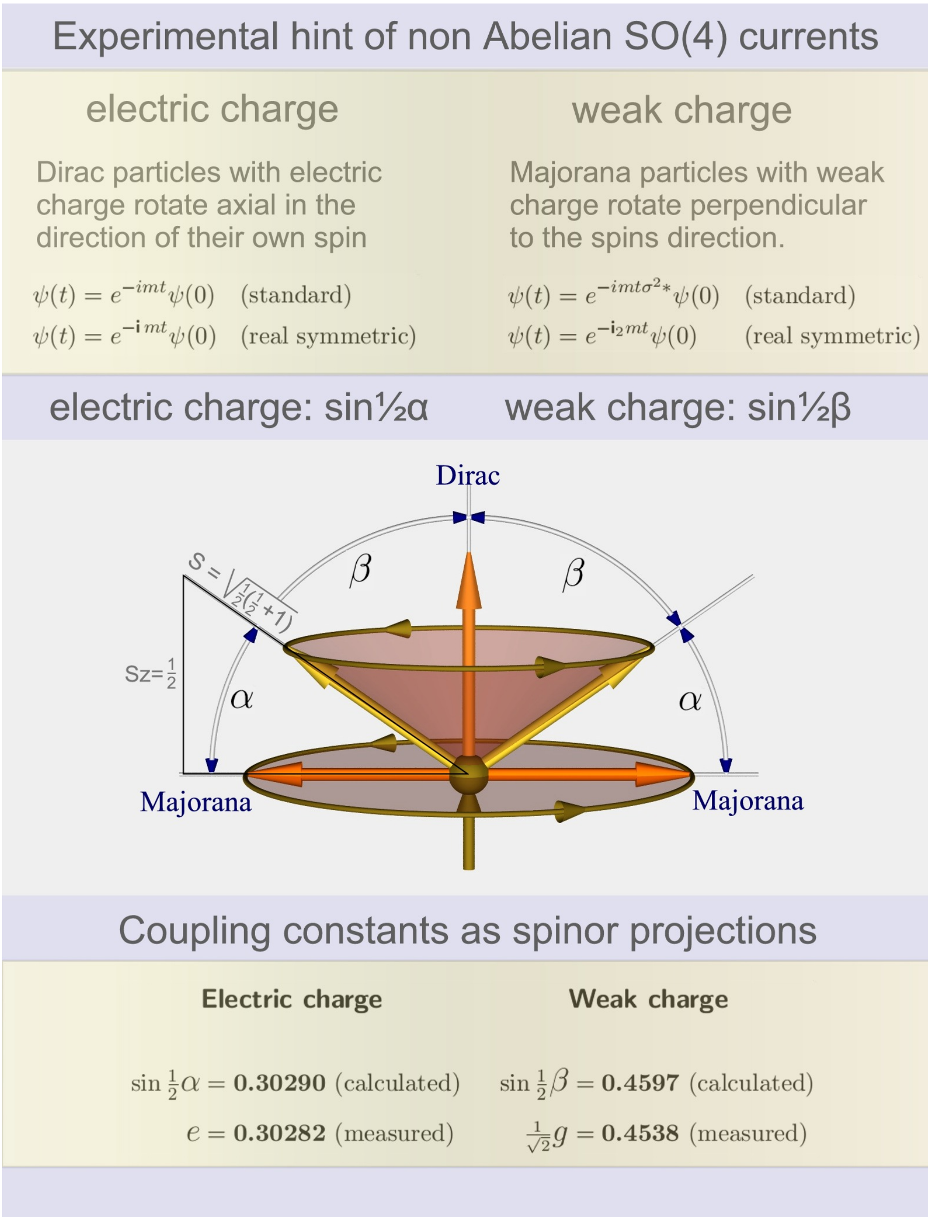

Figure 21: base-vectors as

If we now assume that the physical source process takes place in spinor space then we accordingly must use half angles and may expect that the electric and weak charges (the coupling parameters) are expressed by $\sin\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\beta$ as determined directly by the equations of motion in first approximation. We find:

Electric charge	**Weak charge**
$\sin\frac{1}{2}\alpha = \mathbf{0.30290}$ (calculated)	$\sin\frac{1}{2}\beta = \mathbf{0.4597}$ (calculated)
$e = \mathbf{0.30282}$ (measured)	$\frac{1}{\sqrt{2}}g = \mathbf{0.4538}$ (measured)

(198)

So there is a satisfying correspondence to the measured values of the coupling parameters as they occur in the Lagrangian of the Standard model. The second measured value is calculated from direct measurements.

$$\frac{1}{\sqrt{2}}g \quad = \quad \frac{1}{\sqrt{2}}\frac{e}{\sqrt{1-\dfrac{M_w^2}{M_z^2}}} \tag{199}$$

We can take this dual correspondence as an indication that the three currents j_x^μ, j_y^μ and j_z^μ which arise form the local gauge invariance of the non Abelian SO(4) Lagrangian have a physical relevance and proceed our investigations further along this path.

Note that the overall values that turn up in the Lagrangian are the squares e^2 and $(\frac{1}{\sqrt{2}}g)^2$. It is conventional to use the coupling parameter e once in the efficiency of the process where the current $\bar{\psi}\gamma^\mu\psi$ is the source of the field A^μ (in the fermion vertices) and for the second time in the coupling of A^μ to the current in the Lagrangian. So the overall efficiency is e^2. This is just the classical convention from electromagnetic field theory

Here we find the coupling parameters in spinor space rather then in configuration space and hence the bilinear currents already contain the squares of these parameters. This would mean that the inefficiency in the physical process is entirely determined by the source process $\bar{\psi}\gamma^\mu\psi \rightarrow A^\mu$ while the coupling between A^μ and j^μ as it occurs in the Lagrangian is 100% effective.